Algebra II
Unit 10
Sequences and Series

Unit "I can" statements:

1. I can determine if a sequence is arithmetic, geometric, or neither, and I can extend a sequence.
2. I can find the formula for an arithmetic sequence, find a specified term within an arithmetic sequence, and find a given number of arithmetic means between two terms in an arithmetic sequence.
3. I can find the formula for a geometric sequence, find a specified term within a geometric sequence, and find a given number of geometric means between two terms in a geometric sequence.
4. I can write series in expanded form and in summation (sigma) notation.
5. I can find partial sums of arithmetic and geometric series.

Common Core State Standards that are addressed in this unit include: A.SSE.1a, A.SSE.4b, A.CED.2a
For more information see www.corestandards.org/Math/

Introduction to Sequences

In this, the last unit, we will study sequences and series. Sequences are simply a type of pattern.

Definition: A **Sequence** is a function whose domain is the set of natural (counting) numbers, and whose rage is the set of term values.

Example: Consider the sequence 3, 5, 7, 9, … can be written in a table as

Term number (x)	1	2	3	4	…
Term value (y)	3	5	7	9	…

The notation that is used is this.

$t_1 = 3$

$t_2 = 5$

$t_3 = 7$

$t_4 = 9$

$t_n = 2n+1$

There are many special types of sequences. We will mainly concentrate on two.

Arithmetic Sequence – all terms are separated by a common difference, d.

Example: 7, 11, 15, 19, 23, … $d=4$

Geometric Sequence – all terms are separated by a common ratio, r.

Example 1, 3, 9, 27, … $r=3$

Break for Practice:

1. Identify the following as arithmetic, geometric, or neither. Then fill in the missing terms.

a) 7, 12, 17, 22, 27, 32 arithmetic $d=5$

+5 +5 +5

b) 2, -4, 8, -16, 32, -64 geometric $r=-2$

x-2 x-2 x-2

c) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ neither (denominator is going up by 1)

d) -1, 2, -3, 4, $\underline{-5}$, $\underline{6}$ neither

e) 21, 15, 9, 3, $\underline{-3}$, $\underline{-9}$ arithmetic $d=-6$

-6 -6 -6

f) 5, 15, 45, 135, $\underline{405}$, $\underline{1215}$ geometric $r=3$

×3 ×3 ×3

2. Find the first 4 terms, and identify the sequence as arithmetic, geometric, or neither.

a) $t_n = 1 - 2n$

$t_1 = 1-2(1) = -1$

$t_2 = 1-2(2) = -3$

$t_3 = 1-2(3) = -5$

$t_4 = 1-2(4) = -7$

$-1, -3, -5, -7$

$\boxed{\text{arithmetic}}$ $d=-2$

b) $t_n = 2(3^n)$

$t_1 = 2(3^1) = 6$ $t_3 = 2(3^3) = 54$

$t_2 = 2(3^2) = 18$ $t_4 = 2(3^4) = 162$

$6, 18, 54, 162$

$\boxed{\text{geometric}}$ $r=3$

c) $t_n = \frac{1}{n^2}$

$t_1 = \frac{1}{1^2} = 1$ $t_3 = \frac{1}{3^2} = \frac{1}{9}$

$t_2 = \frac{1}{2^2} = \frac{1}{4}$ $t_4 = \frac{1}{4^2} = \frac{1}{16}$

$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

neither

3. Find the next two terms by looking at the pattern in the difference between terms.

a) 8, 9, 11, 14, $\underline{18}$, $\underline{23}$

+1 +2 +3 +4 +5

b) 5, 7, 11, 17, $\underline{25}$, $\underline{35}$

+2 +4 +6 +8 +10

Extended Practice:

1. Identify the following as arithmetic, geometric, or neither. Then fill in the missing terms.

a) 20, 17,14, 11, 8, 5   arithmetic $d = -3$

b) 5, 9, 13, 17, 21, 25   arithmetic $d = 4$

c) 1, 5, 25, 125, 625, 3125   geometric $r = 5$

d) 256, 64, 16, 4, 1, $\frac{1}{4}$ geometric $r = \frac{1}{4}$

e) 18, 22, 26, 30, 34, 38   arithmetic $d = 4$

f) 4, 0, -4, -8, -12, -16   arithmetic $d = -4$

g) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$, $\frac{1}{25}$, $\frac{1}{36}$ neither $\left(\frac{1}{\text{perfect square}}\right)$

h) 32, -16, 8, -4, 2, -1   geometric $r = -\frac{1}{2}$

2. Find the first four terms of the sequence with the given formula. Then tell whether the sequence is arithmetic, geometric, or neither.

a) $t_n = 4n + 3$

$t_1 = 4(1)+3 = 7$ $t_3 = 4(3)+3 = 15$ 7, 11, 15, 19

$t_2 = 4(2)+3 = 11$ $t_4 = 4(4)+3 = 19$ $\boxed{\text{arithmetic}}$ $d = 4$

b) $t_n = 2n + 1$

$t_1 = 2(1)+1 = 3$ $t_3 = 2(3)+1 = 7$ 3, 5, 7, 9

$t_2 = 2(2)+1 = 5$ $t_4 = 2(4)+1 = 9$ $\boxed{\text{arithmetic}}$ $d = 2$

c) $t_n = 3^{n-1}$

$t_1 = 3^{1-1} = 1$ $t_3 = 3^{3-1} = 9$ 1, 3, 9, 27

$t_2 = 3^{2-1} = 3$ $t_4 = 3^{4-1} = 27$ $\boxed{\text{geometric}}$ $r = 3$

d) $t_n = 2 \cdot 3^n$

$t_1 = 2 \cdot 3^1 = 6$ $t_3 = 2 \cdot 3^3 = 54$ 6, 18, 54, 162

$t_2 = 2 \cdot 3^2 = 18$ $t_4 = 2 \cdot 3^4 = 162$ $\boxed{\text{geometric}}$ $r = 3$

e) $t_n = \frac{(-2)^n}{8}$ $t_1 = \frac{(-2)^1}{8} = \frac{-1}{4}$ $t_3 = \frac{(-2)^3}{8} = -1$

$t_2 = \frac{(-2)^2}{8} = \frac{1}{2}$ $t_4 = \frac{(-2)^4}{8} = 2$ $\boxed{\text{geometric}}$ $r=-2$

3. Find the next two terms of each sequence by using the pattern in the differences between terms.

a) 60, 48, 38, 30, 24, 20, 18

-12 -10 -8 -6 -4 -2

b) 24, 23, 21, 17, 9, -7, -39

-1 -2 -4 -8 -16 -32

c) 1, 3, 7, 15, 31, 63, 127

+2 +4 +8 +16 +32 +64

d) 0, 1, 4, 13, 40, 121, 364

+1 +3 +9 +27 +81 +243

e) 1, 1, 2, 3, 5, 8, 13, 21, 34

+0 +1 +1 +2 +3 +5 +8 +13

add 2 previous terms

Fibonacci Sequence

f) 1, 3, 6, 11, 19, 31, 48, 71 (Hint: Look at the second differences, that is, the differences of the differences between terms.)

+2 +3 +5 +8 +12 +17 +23

+1 +2 +3 +4 +5 +6

Arithmetic Sequences

In this section we shall see how we can write and use formulas for arithmetic sequences.

Consider the sequence $\underbrace{3, 10}_{+7}, \underbrace{17}_{+7}, \underbrace{24}_{+7}, \ldots$ Verify this is arithmetic and identify the common difference.

arithmetic $d = 7$

$t_1 = 3$

$t_2 = 10 = 3 + 7 = 3 + 1(7)$

$t_3 = 17 = 3 + 7 + 7 = 3 + 2(7)$

$t_4 = 24 = 3 + 7 + 7 + 7 = 3 + 3(7)$

$t_n = 3 + (n-1)7$

first term

common difference (d)

Result: For an arithmetic sequence, the formula is $t_n = t_1 + (n-1)d$

Break for Practice:

1. Write a formula for each of the following.

   a) $\underbrace{7, 15}_{+8}, \underbrace{23}_{+8}, \underbrace{31}_{+8}, \ldots$

   $t_1 = 7$

   $d = 8$

   $\boxed{t_n = 7 + (n-1)8}$

   $= 7 + 8n - 8$

   $\boxed{t_n = -1 + 8n}$

   either one

   b) $\underbrace{100, 90}_{-10}, \underbrace{80}_{-10}, \underbrace{70}_{-10}, \ldots$

   $t_1 = 100$

   $d = -10$

   $\boxed{t_n = 100 + (n-1)(-10)}$ use parentheses

   $= 100 - 10n + 10$

   $\boxed{t_n = 110 - 10n}$

   c) $\underbrace{3, 7}_{+4}, \underbrace{11}_{+4}, \underbrace{15}_{+4}, \ldots$

   $t_1 = 3$

   $d = 4$

   $\boxed{t_n = 3 + (n-1)4}$

   $= 3 + 4n - 4$

   $\boxed{t_n = -1 + 4n}$

2. Find the specified term in the Arithmetic Sequence.

   a) $\underbrace{2, 5}_{+3}, 8, \ldots$ $t_{17} =$ 50

   $t_1 = 2$

   $d = 3$

   $n = 17$

   $t_{17} = 2 + (17-1)3$

   $= 2 + (16)3$

   $= 2 + 48$

   $\boxed{t_{17} = 50}$

b) 912, 882, 852, 822, … $\boxed{t_{43} = -348}$

$t_1 = 912$

$d = -30$

$n = 43$

$t_{43} = 912 + (43-1)(-30)$

$= 912 + 42(-30)$

$= 912 - 1260$

$t_{43} = -348$

c) $t_2 = 9$ $t_5 = 21$ $\boxed{t_{41} = 165}$

$t_2 = 9$

$n = 2$

$t_5 = 21$

$n = 5$

$9 = t_1 + (2-1)d$ $= (9 = t_1 + d)-1$ $\Rightarrow$

$21 = t_1 + (5-1)d$ $21 = t_1 + 4d$

$-9 = \cancel{-t_1} - d$

$+\ 21 = \cancel{t_1} + 4d$

$\frac{12}{3} = \frac{3d}{3}$

$4 = d$

$9 = t_1 + 4$

$-4 \quad -4$

$5 = t_1$

$t_{41} = 5 + (41-1)4$

$t_{41} = 5 + 40(4) = 165$

d) $t_{10} = 41$ $t_{15} = 61$ $\boxed{t_3 = 13}$

$t_{10} = 41$

$t_{15} = 61$

$41 = t_1 + (10-1)d$

$61 = t_1 + (15-1)d$

$41 = \cancel{t_1} + 9d$

$-\ 61 = \cancel{t_1} + 14d$

$\frac{-20}{-5} = \frac{-5d}{-5}$

$4 = d$

$41 = t_1 + 9(4)$

$-36 \quad -36$

$5 = t_1$

$t_3 = 5 + (3-1)4$

$5 + 2(4)$

$t_3 = 13$

Extended Practice:

1. Write a formula for each of the following.

a) 24, 32, 40, 48, …

$t_1 = 24$ $d = 8$

$\boxed{t_n = 24 + (n-1)8}$ or $\boxed{t_n = 16 + 8n}$

$= 24 + 8n - 8$

b) 30, 20, 10, 0, …

$t_1 = 30$ $d = -10$

$\boxed{t_n = 30 + (n-1)(-10)}$ or $\boxed{t_n = 40 - 10n}$

$= 30 - 10n + 10$

c) -3, -10, -17, -24, …

$t_1 = -3$ $d = -7$

$\boxed{t_n = -3 + (n-1)(-7)}$ or $\boxed{t_n = 4 - 7n}$

$= -3 - 7n + 7$

d) -6, -1, 4, 9, …

$t_1 = -6$ $d = 5$

$\boxed{t_n = -6 + (n-1)5}$ or $\boxed{t_n = -11 + 5n}$

$-6 + 5n - 5$

e) 7, 11, 15, 19, …

$t_1 = 7$ $d = 4$

$\boxed{t_n = 7 + (n-1)4}$ or $\boxed{t_n = 3 + 4n}$

$= 7 + 4n - 4$

2. Find the specified term of each arithmetic sequence. $t_n = t_1 + (n-1)d$

a) 4, 9, 14, 19, … $\boxed{t_{21} = 104}$

$t_1 = 4$
$d = 5$
$n = 21$

$t_{21} = 4 + (21-1)5$
$= 4 + 20(5)$

b) 3, 11, 19, … $\boxed{t_{31} = 243}$

$t_1 = 3$
$d = 8$
$n = 31$

$t_{31} = 3 + (31-1)(8)$
$= 3 + 30(8)$
$= 3 + 240$

c) 100, 98, 96, … $\boxed{t_{25} = 52}$

$t_1 = 100$
$d = -2$
$n = 25$

$t_{25} = 100 + (25-1)(-2)$
$= 100 + 24(-2)$
$= 100 - 48$

d) 3, 3.5, 4, 4.5, … $\boxed{t_{101} = 53}$

$t_1 = 3$ $d = .5$
$n = 101$

$t_{101} = 3 + (101-1)(.5)$
$= 3 + 100(.5)$
$= 3 + 50$

e) 17, 7, -3, … $\boxed{t_{1000} = -9973}$

$t_1 = 17$
$d = -10$
$n = 1000$

$t_{1000} = 17 + (1000-1)(-10)$
$= 17 + 999(-10)$
$= 17 - 9990$

f) $t_2 = 7$ $t_4 = 8$ $\boxed{t_1 = 6\frac{1}{2}}$

$n=2, t_2 = 7$
$n=4, t_4 = 8$

$7 = t_1 + (2-1)d$
$8 = t_1 + (4-1)d$

$7 = \cancel{t_1} + d$
$-8 = t_1 + 3d$
$-1 = -2d$
$\frac{-1}{-2} = \frac{-2d}{-2}$

$d = \frac{1}{2}$
$7 = t_1 + \frac{1}{2}$
$-\frac{1}{2} \quad -\frac{1}{2}$
$6\frac{1}{2} = t_1$

g) $t_8 = 60$ $t_{12} = 48$ $\boxed{t_{40} = -36}$

$n=8: t_8 = 60$
$n=12: t_{12} = 48$

$60 = t_1 + (8-1)d$
$48 = t_1 + (12-1)d$

$60 = \cancel{t_1} + 7d$
$+(-48 = -\cancel{t_1} - 11d)$
$12 = -4d$
$\frac{12}{-4} = \frac{-4d}{-4}$
$-3 = d$

$60 = t_1 + 7(-3)$
$+21 \quad +21$
$81 = t_1$

$t_{40} = 81 + (40-1)(-3)$
$= 81 + 39(-3)$
$= 81 - 117$
$t_{40} = -36$