Series and Sigma (Summation) Notation

Now that we have spent several days exploring sequences, we are ready to explore the topic of series. First we need to understand what a series is.

Example: Sequence : 1, 3, 5, 7, 9, …

Related Series: 1+ 3 + 5 + 7 + 9 + …

Definition: A **Series** is the sum of the terms in a sequence.

Now, since series have an infinite number of terms, most of them will have an infinite sum. Because of this, it is usually more interesting to consider partial sums.

Definition: A **Partial Sum** is the sum of a finite number of terms in a series.

Notation: S_n stands for the n^{th} partial sum, which is the sum of the first n terms in a series.

Example: Consider 2 + 7 + 12 + 17 + …

$S_1 = 2$

$S_2 = 2+7 = 9$

$S_3 = 2+7+12 = 21$

Since it can take a lot of space and time to write out all of the terms, a shorthand notation was developed. This is called sigma or summation notation.

Sigma or Summation Notation:

* evaluate each term until k=n

$$S_n = \sum_{k=1}^{n} t_k$$

(end) (start)

* t_k is the formula for the terms

Break for Practice:

1. Expand $S_5 = \sum_{k=1}^{5}(5k+3)$ = 8 + 13 + 18 + 23 + 28 = 90

k = 1, 2, 3, 4, 5

$\boxed{S_5 = 90}$

2. Expand $S_4 = \sum_{k=1}^{4} 128\left(\frac{1}{2}\right)^{k-1}$ = 128 + 64 + 32 + 16 = 240

k=1, k=2, k=3, k=4

$\boxed{S_4 = 240}$

Extended Practice: Expand each of the following partial sums.

1. $\sum_{k=1}^{6}(k+10) = \underset{k=1}{11} + \underset{k=2}{12} + \underset{k=3}{13} + \underset{k=4}{14} + \underset{k=5}{15} + \underset{k=6}{16} = \boxed{81}$

2. $\sum_{k=1}^{8} 3k = \underset{k=1}{3} + \underset{k=2}{6} + \underset{k=3}{9} + \underset{k=4}{12} + \underset{k=5}{15} + \underset{k=6}{18} + \underset{k=7}{21} + \underset{k=8}{24} = \boxed{108}$

3. $\sum_{k=1}^{6} 2^k = \underset{k=1}{2} + \underset{k=2}{4} + \underset{k=3}{8} + \underset{k=4}{16} + \underset{k=5}{32} + \underset{k=6}{64} = \boxed{126}$

4. $\sum_{k=4}^{10}(3k-2) = \overset{12-2}{\underset{k=4}{10}} + \overset{15-2}{\underset{k=5}{13}} + \overset{18-2}{\underset{k=6}{16}} + \overset{21-2}{\underset{k=7}{19}} + \overset{24-2}{\underset{k=8}{22}} + \overset{27-2}{\underset{k=9}{25}} + \overset{30-2}{\underset{k=10}{28}} = \boxed{133}$

5. $\sum_{k=0}^{5}\frac{(-1)^k}{k+1} = \underset{k=0}{1} + \underset{k=1}{\left(-\frac{1}{2}\right)} + \underset{k=2}{\frac{1}{3}} + \underset{k=3}{\left(-\frac{1}{4}\right)} + \underset{k=4}{\frac{1}{5}} + \underset{k=5}{\left(-\frac{1}{6}\right)} = \boxed{\frac{37}{60}}$ or $.6167$

6. $\sum_{k=0}^{3} 4^{-k} = \underset{k=0}{1} + \underset{k=1}{\frac{1}{4}} + \underset{k=2}{\frac{1}{16}} + \underset{k=3}{\frac{1}{64}} = \boxed{\frac{85}{64}}$ or 1.328

7. $\sum_{k=3}^{8}|5-k| = \overset{|2|}{\underset{k=3}{2}} + \overset{|1|}{\underset{k=4}{1}} + \overset{|0|}{\underset{k=5}{0}} + \overset{|-1|}{\underset{k=6}{1}} + \overset{|-2|}{\underset{k=7}{2}} + \overset{|-3|}{\underset{k=8}{3}} = \boxed{9}$

8. $\sum_{k=1}^{4}(-k)^{k+1} = \underset{k=1}{1} + \underset{k=2}{(-8)} + \underset{k=3}{81} + \underset{k=4}{(-1024)} = \boxed{-950}$

Now we will try to go in the reverse direction. We will rewrite a series from expanded form into sigma notation. It will be useful on many of the problems to remember the formulas for arithmetic and geometric sequences.

Review: **Arithmetic Sequence** formula: $t_n = t_1 + (n-1)d$

Geometric Sequence formula: $t_n = t_1 \cdot r^{(n-1)}$

Break for Practice: Rewrite each series into sigma notation.

1. 3 + 10 + 17 + 24 + … + 66 arithmetic

$t_1 = 3$, $d = 7$, $t_n = 66$

$66 = 3 + (n-1)7$

$-3 \quad -3$

$63 = (n-1)7$

$\frac{63}{7} = \frac{(n-1)7}{7}$

$9 = n-1$

$+1 \quad +1$

$\boxed{10 = n}$

$\sum_{k=1}^{10} (3 + (k-1)7)$ simplify $\boxed{\sum_{k=1}^{10} (-4 + 7k)}$

$t_n = 3 + (n-1)7$

or

$= 3 + 7n - 7 = -4 + 7n$

2. 3 + 12 + 48 + 192 + … + 12,582,912

$t_1 = 3$

$r = 4$

$t_n = 12,582,912$

$n = ?$

$\frac{12,582,912}{3} = \frac{3(4)^{n-1}}{3}$

$4194304 = 4^{n-1}$

$\log_4 4194304 = n-1$

$\frac{\log 4194304}{\log 4} = n-1$

$11 = n-1$

$11 = n-1$

$+1 \quad +1$

$12 = n$

$t_n = 3(4)^{n-1}$

$\boxed{\sum_{k=1}^{12} \left(3(4)^{k-1}\right)}$

3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{17}$

$k=1 \quad k=2 \quad k=3 \quad k=4 \ldots \quad k=17$

$\boxed{\sum_{k=1}^{17} \left(\frac{1}{k}\right)}$

4. 3 + 9 + 27 + 81 + …

$t_1 = 3$

$r = 3$

$n = \infty$

$\boxed{\sum_{k=1}^{\infty} \left(3(3)^{k-1}\right)} = \boxed{\sum_{k=1}^{\infty} 3^k}$ simplified

$3^1 \cdot 3^{k-1} = 3^k$

5. -5 + 10 – 20 + 40 - …

$t_1 = -5$

$r = -2$

$n = \infty$

$$\boxed{\sum_{k=1}^{\infty}\left(-5(-2)^{k-1}\right)}$$

without (-1)

6. -6 + 10 – 14 + 18 – 22 + … $\rightarrow 6 + 10 + 14 + 18 + 22$

k=1 k=2 k=3 k=4 k=5

$t_1 = 6$

$d = 4$ and $r = -1$

$n = \infty$

$$\boxed{\sum_{k=1}^{\infty}\left[(-1)^k(4k+2)\right]}$$

$t_n = 6 + (n-1)4$

Not geometric nor arithmetic overall but parts yes

$t_n = 6 + 4n - 4$

$t_n = 2 + 4n = 4n + 2$

Extended Practice: Rewrite each series into sigma notation.

1. 2 + 4 + 6 + …+ 1000

$t_1 = 2$ $n = ?$

$d = 2$

$t_n = 1000$

$1000 = 2 + (n-1)2$

$-2 \quad -2$

$\frac{998}{2} = \frac{(n-1)2}{2}$

$499 = n - 1$

$+1 \quad +1$

$500 = n$

$$\boxed{\sum_{k=1}^{500}(2k)}$$

$t_n = 2 + (n-1)2$

$t_n = 2n$

2. 5 + 10 + 15 + … + 250

$t_1 = 5$

$d = 5$

$t_n = 250$

$n = ?$

$250 = 5 + (n-1)5$

$-5 \quad -5$

$\frac{245}{5} = \frac{(n-1)5}{5}$

$49 = (n-1)$

$+1 \quad +1$

$\rightarrow n = 50$

$t_n = 5 + (n-1)5$

$5 + 5n - 5$

$t_n = 5n$

$$\boxed{\sum_{k=1}^{50} 5k}$$

3. $1^3 + 2^3 + 3^3 + \cdots + 20^3$

k=1 k=2 k=3 k=20

$$\boxed{\sum_{k=1}^{20}(k^3)}$$

$n = 20$

4. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{99}{100}$

$k=1 \quad k=2 \quad k=3 \quad k=99$

$n=99$

$$\boxed{\sum_{k=1}^{99}\left(\frac{k}{k+1}\right)}$$

5. 3 + 7 + 11 + 15 + … + 399

$t_1=3$

$d=4$

$t_n=399$

$n=?$

$399=3+(n-1)4$

$-3 \quad -3$

$396=(n-1)4$

$\frac{396}{4}=\frac{(n-1)4}{4}$

$99=n-1$

$+1 \quad +1$

$100=n$

$t_n=3+(n-1)4$

$3+4n-4$

$t_n=4n-1$

$$\boxed{\sum_{k=1}^{100}(4k-1)}$$

6. 1 + 2 + 4 + 8 + … + 64

$t_1=1$

$r=2$

$t_n=64$

$n=?$

$64=1(2)^{n-1}$

$64=2^{(n-1)}$

$\log_2 64=n-1$

$\frac{\log 64}{\log 2}=n-1$

$6=n-1$

$+1 \quad +1$

$7=n$

$t_n=1(2)^{n-1}$

$t_n=2^{n-1}$

$$\boxed{\sum_{k=1}^{7}\left(2^{k-1}\right)}$$

7. $-9+3-1+\frac{1}{3}-\cdots$

$n=\infty$

$t_1=-9$

$r=\frac{1}{-3}$

$t_n=-9\left(\frac{1}{-3}\right)^{n-1}$

$$\boxed{\sum_{k=1}^{\infty}\left(-9\left(\frac{1}{-3}\right)^{k-1}\right)}$$

8. 8 – 4 + 2 – 1 + …

$n=\infty$

$t_1=8$

$r=\frac{1}{-2}$

$t_n=8\left(-\frac{1}{2}\right)^{n-1}$

$$\boxed{\sum_{k=1}^{\infty}\left[8\left(-\frac{1}{2}\right)^{k-1}\right]}$$

9. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$

$k=1 \quad k=2 \quad k=3 \quad k=4$

$t_1=1$

$n=\infty$

$t_k=\frac{1}{k^2}$

$$\boxed{\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)}$$