Algebra II
Unit 6
Irrational and Complex Numbers

Unit "I can" statements:

1. I can find roots of real numbers.
2. I can simplify expressions involving radicals.
3. I can simplify expressions involving sums of radicals.
4. I can simplify products and quotients of binomials that contain radicals.
5. I can solve equations containing radicals.
6. I can find and use decimal representations of real numbers.
7. I can use imaginary numbers to simplify square roots of negative numbers.
8. I can add, subtract, multiply, and divide complex numbers.

Common Core State Standards that are addressed in this unit include: A.SSE.2a, A.CED.1a, A.REI.2, A.REI.2a, N.CN.1a, N.CN.2a, N.CN.3, N.CN.7c
For more information see www.corestandards.org/Math/

Roots of Real Numbers

In this unit we will be exploring roots and radicals. We will begin with the ones you are familiar with, and extend the ideas from there.

Square Roots are solutions to the equation $x^2 = b$. Every positive Number b has 2 square roots, $\sqrt{b}$ and $-\sqrt{b}$. The positive square root, $\sqrt{b}$, is Known as the principle square root.

(Note: Since the square of a real number is never negative, the equation $x^2 = b$ has no real solution if $b < 0$.)

Break for Practice: Simplify

1. $\sqrt{25} = 5$
2. $-\sqrt{36} = -6$
3. $\sqrt{0.81} = .9$
4. $\sqrt{-25}$

   no real solution
5. $\sqrt{\frac{4}{49}} = \frac{2}{7}$

   $\frac{\sqrt{4}}{\sqrt{49}}$

Solve each equation.

6. $\sqrt{x^2} = \sqrt{16}$

   $x = \pm 4$
7. $x^2 - 9 = 0$

   $+9 \quad +9$

   $\sqrt{x^2} = \sqrt{9}$ $\quad x = \pm 3$
8. $x^2 + 25 = 0$

   $-25 \quad -25$

   $\sqrt{x^2} = \sqrt{-25}$

   no real solution
10. $3x^2 - 21 = 0$

   $+21 \quad +21$

   $\frac{3x^2}{3} = \frac{21}{3}$ $\quad \sqrt{x^2} = \sqrt{7}$ $\quad x = \pm\sqrt{7}$

Cube Roots are solutions to the equation $x^3 = b$. Every real number has exactly one real cube root, $\sqrt[3]{b}$.

Break for Practice: Simplify

1. $\sqrt[3]{64} = 4$
2. $\sqrt[3]{0} = 0$
3. $\sqrt[3]{-27} = -3$

Summary: In general we can write the following.

1. An n^{th} root of b is a solution of the equation $x^n = b$.
2. a) If n is even and $b > 0$, there are two real n^{th} roots.

   The principal n^{th} root is written $\sqrt[n]{b}$.
   The other n^{th} root is written $-\sqrt[n]{b}$.

   b) If n is even and $b = 0$, there is one n^{th} root: $\sqrt[n]{0} = 0$

   c) If n is even and $b < 0$, there is no real n^{th} root.
3. If n is odd, there is exactly one real n^{th} root.

Notation: Each part of the notation has a special name.

index → $\sqrt[n]{b}$ ← radicand

radical sign

Break for Practice: Simplify

1. $\sqrt{100} = 10$
2. $-\sqrt{100} = -10$
3. $\sqrt[4]{-100}$ no real solution
4. $\sqrt[4]{0.0081} = .3$
5. $\sqrt{(-8)^2}$ $\sqrt{64} = 8$
6. $\sqrt{-8^2} = \sqrt{-64}$ no real solution
7. $\sqrt[3]{\frac{-27}{64}} = \frac{-3}{4}$
8. $\frac{2}{-\sqrt{36}} = \frac{2}{-6} = -\frac{1}{3}$
9. $\sqrt[5]{32} = 2$

10. For what values of x does each expression represent a real number?

a) $\sqrt{x+2}$

$x + 2 \geq 0$
$-2 \quad -2$

$x \geq -2$

b) $\sqrt{3-x}$

$3 - x \geq 0$
$-3 \quad -3$

$\frac{-x}{-1} \geq \frac{-3}{-1}$

$x \leq 3$

c) $\sqrt[3]{x}$

$\boxed{\text{All } \mathbb{R}\text{'s}}$

Extended Practice:

1. Simplify each expression. If the expression does not represent a real number, say so.

$\sqrt{16}$ = 4	$-\sqrt{16}$ = −4	$\sqrt{-16}$ no real solution	$\sqrt[4]{16}$ = 2
$\sqrt{64}$ = 8	$\sqrt{-64}$ = no real solution	$\sqrt[3]{64}$ = 4	$\sqrt[3]{-64}$ = −4
$\sqrt{0.01}$ = .1	$\sqrt{-0.01}$ No real solution	$\sqrt[3]{0.001}$ = .1	$\sqrt[3]{-0.001}$ = −.1
$\sqrt{0.04}$ = .2	$-\sqrt{0.04}$ = −.2	$\sqrt{0.0004}$ = .02	$\sqrt{-0.0004}$ no real solution
$\sqrt{7^2}$ = 7	$\sqrt[3]{7^3}$ = 7	$\sqrt[4]{(-7)^4}$ = +7	$\sqrt[5]{(-7)^5}$ = −7
$\sqrt{\frac{1}{64}} = \frac{1}{8}$	$\frac{1}{\sqrt{64}} = \frac{1}{8}$	$\sqrt[3]{-\frac{1}{64}} = \frac{-1}{4}$	$-\frac{1}{\sqrt[3]{64}}$ $-\frac{1}{4}$
$\sqrt{\frac{1}{16}} = \frac{1}{4}$	$\sqrt{\frac{81}{16}} = \frac{9}{4}$	$\sqrt[4]{\frac{1}{16}} = \frac{1}{2}$	$\sqrt[4]{\frac{81}{16}} = \frac{3}{2}$

2. Find the real roots of each equation. If there are none, say so.

a) $\sqrt{x^2} = \sqrt{144}$

$x = \pm 12$

b) $y^2 - 7 = 0$

$+7 \quad +7$

$\sqrt{y^2} = \sqrt{7}$ $\quad \boxed{y = \pm\sqrt{7}}$

c) $16y^2 = 25$

$\frac{16y^2}{16} = \frac{25}{16}$

$\sqrt{y^2} = \sqrt{\frac{25}{16}}$ $\quad \boxed{y = \pm\frac{5}{4}}$

d) $0 = 4 + 16x^2$

$-4 \quad -4$

$\frac{-4}{16} = \frac{16x^2}{16}$ $= \sqrt{\frac{-1}{4}} = \sqrt{x^2}$

$\boxed{\text{no real Solution}}$

3. For what values of x does each expression represent a real number?

a) $\sqrt{x+1}$

$x + 1 \geq 0$

$-1 \quad -1$

$x \geq -1$

b) $\sqrt{x-1}$

$x - 1 \geq 0$

$+1 \quad +1$

$x \geq 1$

c) $\sqrt[3]{x-1}$

All $\mathbb{R}$'s

Simplifying Radicals

One of the first skills that is needed with radicals is knowing how to simplify radicals.

Rules: An expression containing n^{th} roots is in simplest radical form if:

a) no radicand contains a factor (other than 1) that is a perfect n^{th} power.

b) Every denominator has been rationalized, so that no radicand is a fraction and no radical is in the denominator.

We will begin with the first rule, and we will start with square roots since you may have had some experience with those in the past.

Break for Practice: Simplify

Know:	
$2^2 =$	4
$3^2 =$	9
$4^2 =$	16
$5^2 =$	25
$6^2 =$	36
$7^2 =$	49
$8^2 =$	64
$9^2 =$	81
$10^2 =$	100
$11^2 =$	121
$12^2 =$	144

1. $\sqrt{125} = \sqrt{25} \cdot \sqrt{5}$
   $\boxed{5\sqrt{5}}$

2. $\sqrt{72} = \sqrt{36} \cdot \sqrt{2}$
   $\boxed{6\sqrt{2}}$

3. $\sqrt{72}$ (no this isn't a typo, I want you to see something)
   $\sqrt{9}\ \sqrt{8}$
   $3 \cdot \sqrt{4}\sqrt{2}$
   $3 \cdot 2\sqrt{2} = \boxed{6\sqrt{2}}$

4. $\sqrt{12} = \sqrt{4} \cdot \sqrt{3}$
   $\boxed{2\sqrt{3}}$

5. $\sqrt{48} = \sqrt{16} \cdot \sqrt{3}$
   $\boxed{4\sqrt{3}}$

Other radicals are more difficult since we are not as familiar with other powers than 2. For this reason, we will use a factor tree method.

Break for Practice: Simplify

1. $\sqrt[5]{64}$

32 2
8 4
4 2 2 2
2 2

$64 = 2^6 \quad \sqrt[5]{64} = \sqrt[5]{2^6} = \sqrt[5]{2^5 \cdot 2^1}$

# of 2's that come out

$5\overline{)6}$ → 1
-5
1 ← # of 2's left behind under radical

$= 2\sqrt[5]{2}$

2. $\sqrt[3]{1250}$ $= \sqrt[3]{5^4 \cdot 2}$

125 10

25 5 2 5

5 5

$5\sqrt[3]{5 \cdot 2}$

$\boxed{5\sqrt[3]{10}}$

3. $\sqrt[3]{640}$

64 10

8 8 2 5

4 2 4 2

2 2 2 2

$\sqrt[3]{2^7 \cdot 5}$

$2^2\sqrt[3]{2 \cdot 5}$

$\boxed{4\sqrt[3]{10}}$

4. $\sqrt[3]{3125}$

5 625

25 25

5 5 5 5

$\sqrt[3]{5^5}$

$5\sqrt[3]{5^2}$

$\boxed{5\sqrt[3]{25}}$

5. $\sqrt[3]{5^4 \cdot 2^5 a^4 b^3}$

$5 \cdot 2ab\sqrt[3]{5 \cdot 2^2 \cdot a}$

$\boxed{10ab\sqrt[3]{20a}}$

6. $\sqrt[5]{3^{12}x^2y^6z^5}$

$3^2yz\sqrt[5]{3^2x^2y}$

$\boxed{9yz\sqrt[5]{9x^2y}}$

Extended Practice: Simplify

1. $\sqrt{52}$ $\sqrt{4}\sqrt{13}$ $\boxed{2\sqrt{13}}$	2. $\sqrt{75}$ $\sqrt{25}\sqrt{3}$ $\boxed{5\sqrt{3}}$
3. $\sqrt{108}$ $\sqrt{36}\sqrt{3}$ $\boxed{6\sqrt{3}}$	4. $\sqrt{500}$ $\sqrt{100}\sqrt{5}$ $\boxed{10\sqrt{5}}$
5. $\sqrt[3]{80} = \sqrt[3]{2^4\cdot 5}$ 5 16 4 4 2 2 2 2 $2\sqrt[3]{2\cdot 5}$ $\boxed{2\sqrt[3]{10}}$	6. $\sqrt[3]{486} \to \sqrt[3]{2\cdot 3^5}$ 2 243 9 27 3 3 3 9 3 3 $3\sqrt[3]{2\cdot 3^2}$ $\boxed{3\sqrt[3]{18}}$
7. $\sqrt[4]{486} \to \sqrt[4]{2\cdot 3^5}$ 2 243 9 27 3 3 3 9 3 3 $3\sqrt[4]{2\cdot 3}$ $\boxed{3\sqrt[4]{6}}$	8. $\sqrt[3]{96}$ 3 32 4 8 2 2 2 2 2 $\sqrt[3]{2^5\cdot 3}$ $2\sqrt[3]{2^2\cdot 3} \to \boxed{2\sqrt[3]{12}}$
9. $\sqrt[3]{216} \to \sqrt[3]{2^3\cdot 3^3}$ 6 36 2 3 6 6 2 3 2 3 $2\cdot 3 = \boxed{6}$	10. $\sqrt[4]{3125} \to \sqrt[4]{5^5}$ 5 625 25 25 5 5 5 5 $\boxed{5\sqrt[4]{5}}$
11. $\sqrt[3]{2^7a^5b}$ $2^2a\sqrt[3]{2a^2b}$ $\boxed{4a\sqrt[3]{2a^2b}}$	12. $\sqrt[5]{3^{15}xy^6}$ $3^3y\sqrt[5]{xy}$ $\boxed{27y\sqrt[5]{xy}}$
13. $\sqrt{2^5x^3y^2z}$ $2^2xy\sqrt{2xz}$ $\boxed{4xy\sqrt{2xz}}$	14. $\sqrt[3]{2^5\cdot 3^4ab^6}$ $2\cdot 3\cdot b^2\sqrt[3]{2^2\cdot 3a}$ $\boxed{6b^2\sqrt[3]{12a}}$

Now before addressing the second rule of simplifying radicals, it is necessary to understand how to multiply radicals. The radicals should have the same index, and then the radicands are simply multiplied together.

Property: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Break for Practice: Simplify

1. $\sqrt{2} \cdot \sqrt{6} = \sqrt{12}$
$\sqrt{4}\sqrt{3}$
$\boxed{2\sqrt{3}}$

2. $\sqrt{10} \cdot \sqrt{15} = \sqrt{150}$
$\sqrt{25}\sqrt{6}$
$\boxed{5\sqrt{6}}$

3. $\sqrt[3]{15} \cdot \sqrt[3]{18}$
$\sqrt[3]{270} \to \sqrt[3]{3^3 \cdot 2 \cdot 5} = \boxed{3\sqrt[3]{10}}$
270
3 90
3 30
6 5
2 3

4. $\sqrt[3]{6} \cdot \sqrt[3]{20}$
$\sqrt[3]{120} \to \sqrt[3]{2^3 \cdot 3 \cdot 5}$
30 4
6 5 2 2
2 3
$\boxed{2\sqrt[3]{15}}$

5. $(5\sqrt{6})^2$
$5^2(\sqrt{6})^2$
$25 \cdot 6 = 150$

6. $(4\sqrt{3})^2$
$4^2(\sqrt{3})^2 = 48$
$16 \cdot 3$

At this point we are ready to deal with the second rule for simplifying radicals. Remember that this rule states that every denominator has been rationalized, so that no radicand is a fraction and no radical is in the denominator. Once again it is easiest to deal with square roots first, and then we will extend the technique to other radicals.

Break for Practice: Simplify

1. $\frac{2}{\sqrt{5}} \cdot \left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \boxed{\frac{2\sqrt{5}}{5}}$

2. $\frac{15}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{15\sqrt{3}}{3} = \boxed{5\sqrt{3}}$

3. $\frac{3}{\sqrt[3]{4}}\left(\frac{\sqrt[3]{2}}{\sqrt[3]{2}}\right) = \boxed{\frac{3\sqrt[3]{2}}{2}}$
$\sqrt[3]{2^2} \cdot \sqrt[3]{2}$
$\sqrt[3]{2^3} = 2$

* goal is to get the power to match the index