Algebra II
Unit 7
Quadratic Equations and Functions

Unit "I can" statements:

1. I can solve quadratic equations by completing the square.
2. I can solve quadratic equations and applications by using the quadratic formula.
3. I can use the discriminant to determine the nature of the roots of a quadratic equation.
4. I can graph quadratic functions by rewriting the equations into vertex form.
5. I can write the equation for a parabola when given three points on the parabola.
6. I can solve quadratic function applications.

Common Core State Standards that are addressed in this unit include: N.CM.7c, A.CED.1a, A.CED.2a, A.CED.3a, F.IF.4, F.IF.5, F.IF.7c, F.IF.8
For more information see www.corestandards.org/Math/

Completing the Square

In this unit, we will be investigating quadratic equations and functions. Just like all linear functions could be written in the form $y = mx + b$, all quadratic functions can be written in the form $y = ax^2 + bx + c$, *where* $a, b,$ *and* $c \in$ *reals*. In this unit, we will learn how to solve, graph, and apply quadratic functions.

The first technique that we will learn in this unit for solving quadratics is called completing the square. First we should consider quadratics that you already know how to solve.

Break for Practice: Solve

$\sqrt{x^2} = \sqrt{25}$

$x = \pm 5$

$\sqrt{x^2} = \sqrt{12}$

$x = \pm\sqrt{12}$

12

$4 \quad 3$

$2 \quad 2$

$\boxed{x = \pm 2\sqrt{3}}$

$\sqrt{(x-2)^2} = \sqrt{5}$

$x - 2 = \pm\sqrt{5}$

$+2 \quad +2$

$\boxed{x = 2 \pm \sqrt{5}}$

$\sqrt{(5x+4)^2} = \sqrt{-36}$

$5x + 4 = \pm i\sqrt{36}$

$5x + 4 = \pm 6i$

$-4 \quad -4$

$\frac{5x}{5} = \frac{-4 \pm 6i}{5}$

$\boxed{x = \frac{-4}{5} \pm \frac{6i}{5}}$

If we can transform a quadratic into the above form, then we can solve it. This needed technique is called completing the square.

Example: Solve by completing the square.

$x^2 + 12x - 45 = 0$

$+45 \quad +45$

$x^2 + 12x + 36 = 45 + 36$ $\quad \left(\frac{12}{2}\right)^2$

$\sqrt{(x+6)^2} = \sqrt{81}$ $\quad (6)^2 = 36$

$x + 6 = \pm 9$

$-6 \quad -6$

$x = -6 \pm 9$ $\quad -6+9=3$, $\quad -6-9=-15$

$x = -15, 3$

Steps:

1. Isolate the constant on one side.
2. Divide through by the coefficient of x^2 and add a blank to both sides.
3. Add to both sides (fill in the blanks) the square of half of the coefficient of x. $ax^2 + bx = c$ $\quad \left(\frac{b}{2}\right)^2 = (8)^2$
4. Write in the form $(x \pm q)^2 = r$. * don't forget ±
5. Solve.

Note: The Ancient Babylonians knew how to do all of this with pictures!

Break for Practice: Solve by completing the square.

1. $2x^2 - 8x - 12 = 0$

$+12 \quad +12$

$\frac{2x^2}{2} - \frac{8x}{2} = \frac{12}{2}$ $\quad * \left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$

$x^2 - 4x + 4 = 6 + 4$

$\sqrt{(x-2)^2} = \sqrt{10}$

$x - 2 = \pm\sqrt{10}$

$+2 \quad +2$

$x = 2 \pm \sqrt{10}$ or $x = 2+\sqrt{10}$, $x = 2-\sqrt{10}$

2. $3x^2 - 10 = 12x$

$-12x \quad +10 \quad -12x \quad +10$

$\frac{3x^2}{3} - \frac{12x}{3} = \frac{10}{3}$ $\quad \left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$

$x^2 - 4x + 4 = \frac{10}{3} + 4$ $\quad \frac{10}{3} + \frac{12}{3} = \frac{22}{3}$

$\sqrt{(x-2)^2} = \sqrt{\frac{22}{3}}$

$x - 2 = \pm\frac{\sqrt{22}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$x - 2 = \pm\frac{\sqrt{66}}{3}$

$+2 \quad +2$

$x = 2 \pm \frac{\sqrt{66}}{3}$ or $x = 2 + \frac{\sqrt{66}}{3}$, $x = 2 - \frac{\sqrt{66}}{3}$

Extended Practice:

1. Solve each equation.

$\sqrt{x^2} = \sqrt{3}$ $x = \pm\sqrt{3}$	$\sqrt{(x-1)^2} = \sqrt{3}$ $x - 1 = \pm\sqrt{3}$ $+1 \quad +1$ $x = 1 \pm \sqrt{3}$	$\sqrt{(2x-1)^2} = \sqrt{3}$ $2x - 1 = \pm\sqrt{3}$ $+1 \quad +1$ $\frac{2x}{2} = \frac{1 \pm \sqrt{3}}{2}$ $x = \frac{1 \pm \sqrt{3}}{2}$
$\sqrt{x^2} = \sqrt{-4}$ $i\sqrt{4}$ $x = \pm 2i$	$\sqrt{(x+7)^2} = \sqrt{-4}$ $i\sqrt{4}$ $x + 7 = \pm 2i$ $-7 \quad -7$ $x = -7 \pm 2i$	$\sqrt{(2x+7)^2} = \sqrt{-4}$ $2x + 7 = \pm 2i$ $-7 \quad -7$ $\frac{2x}{2} = \frac{-7}{2} \pm \frac{2i}{2}$ $x = \frac{-7}{2} \pm i$

2. Solve by completing the square.

a) $x^2 - 2x - 5 = 0$

$+5 \quad +5$

$x^2 - 2x \underline{+1} = 5 \underline{+1}$

$\sqrt{(x-1)^2} = \sqrt{6}$

$x - 1 = \pm\sqrt{6}$

$+1 \quad +1$

$x = 1 \pm \sqrt{6}$

$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

b) $y^2 + 6y - 2 = 0$

$+2 \quad +2$

$y^2 + 6y \underline{+9} = 2 \underline{+9}$

$\sqrt{(y+3)^2} = \sqrt{11}$

$y + 3 = \pm\sqrt{11}$

$-3 \quad -3$

$y = -3 \pm \sqrt{11}$

$\left(\frac{6}{2}\right)^2 = (3)^2 = 9$

c) $t^2 + 8 = 4t$

$-4t \quad -8 \quad -4t \quad -8$

$t^2 - 4t \underline{+4} = -8 \underline{+4}$ $\qquad \left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$

$\sqrt{(t-2)^2} = \sqrt{-4}$

$t - 2 = \pm 2i$

$+2 \quad +2$

$\boxed{t = 2 \pm 2i}$

d) $\frac{3n^2}{3} + \frac{12n}{3} = \frac{-1}{3}$

$n^2 + 4n \underline{+4} = \frac{-1}{3} \underline{+4}$ $\qquad \left(\frac{4}{2}\right)^2 = (2)^2 = 4$

$(n+2)^2 = \frac{-1}{3} + \frac{12}{3}$

$\sqrt{(n+2)^2} = \sqrt{\frac{11}{3}}$

$n + 2 = \pm\frac{\sqrt{11}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$n + 2 = \pm\frac{\sqrt{33}}{3}$

$-2 \quad -2$

$\boxed{n = -2 \pm \frac{\sqrt{33}}{3}}$

The Quadratic Formula

In this section we will learn a more efficient method for solving quadratic equations. It is a formula derived from the method of completing the square.

Derivation: Begin with the general form $ax^2 + bx + c = 0$, then complete the square.

$-c \quad -c$

$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$

$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c \cdot 4a}{a \cdot 4a} + \frac{b^2}{4a^2}$

$\left(\frac{b}{a} \div 2\right)^2$

$\left(\frac{b}{a} \cdot \frac{1}{2}\right)^2$

$\left(\frac{b}{2a}\right)^2$

$\frac{b^2}{4a^2}$

$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$

$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$

$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$

$-\frac{b}{2a} \quad -\frac{b}{2a}$

$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$*\quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

quadratic formula

Result: Quadratic Formula: If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note – In order to use this formula, the problem needs to be in the form $ax^2 + bx + c = 0$.

Break for Practice: Solve each equation by using the quadratic formula.

1. $2x^2 - 3x - 5 = 0$

$a = 2$, $b = -3$, $c = -5$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{9 + 40}}{4}$$

$$x = \frac{3 \pm 7}{4}$$

$$\frac{3+7}{4} = \frac{10}{4} = \frac{5}{2}$$

$$\frac{3-7}{4} = \frac{-4}{4} = -1$$

$$x = -1, \frac{5}{2}$$

2. $k^2 - 6k - 1 = 0$

$a = 1$, $b = -6$, $c = -1$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 + 4}}{2}$$

$$= \frac{6 \pm \sqrt{40}}{2}$$

$\sqrt{40} = 2\sqrt{10}$ (4 · 10, 2 · 2 · 5)

$$x = \frac{6 \pm 2\sqrt{10}}{2}$$

$$x = 3 \pm \sqrt{10}$$

3. $3x^2 = -2x - 1$

$+2x + 1 \quad +2x + 1$

$3x^2 + 2x + 1 = 0$

$a = 3$, $b = 2$, $c = 1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

$$x = \frac{-2 \pm \sqrt{-8}}{6}$$

$\sqrt{-8} = 2i\sqrt{2}$ ($i\sqrt{8}$, 4 · 2)

$$x = \frac{-2 \pm 2i\sqrt{2}}{6}$$

$$x = \frac{-2}{6} \pm \frac{2i\sqrt{2}}{6} = \frac{-1}{3} \pm \frac{i\sqrt{2}}{3}$$

4. $(3x - 1)(x + 4) = -9$

$3x^2 + 12x - x - 4 = -9$

$+9 \quad +9$

$3x^2 + 12x - x + 5 = 0$

$3x^2 + 11x + 5 = 0$

$a = 3 \quad b = 11 \quad c = 5$

$$x = \frac{-11 \pm \sqrt{(11)^2 - 4(3)(5)}}{2(3)}$$

$$= \frac{-11 \pm \sqrt{121 - 60}}{6}$$

$$x = \frac{-11 \pm \sqrt{61}}{6}$$

Extended Practice: Solve each equation using the quadratic formula. Give answers involving radicals in simplest radical form.

1. $5k^2 + 3k - 2 = 0$

$a = 5$, $b = 3$, $c = -2$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-2)}}{2(5)}$$

$$= \frac{-3 \pm \sqrt{9 + 40}}{10}$$

$$= \frac{-3 \pm 7}{10}$$

$$x = \frac{-3+7}{10} = \frac{4}{10} = \frac{2}{5}$$

$$x = \frac{-3-7}{10} = \frac{-10}{10} = -1$$

$$\boxed{x = -1, \frac{2}{5}}$$

2. $2p^2 - 3p - 2 = 0$

$a = 2$, $b = -3$, $c = -2$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9 + 16}}{4}$$

$$x = \frac{3 \pm 5}{4} \rightarrow \frac{3+5}{4} = \frac{8}{4} = 2$$

$$\frac{3-5}{4} = \frac{-2}{4} = \frac{-1}{2}$$

$$\boxed{x = 2, -\frac{1}{2}}$$

3. $5r^2 + 8 = -12r$

$+12r \quad +12r$

$5r^2 + 12r + 8 = 0$

$a = 5$, $b = 12$, $c = 8$

$$x = \frac{-12 \pm \sqrt{12^2 - 4(5)(8)}}{2(5)}$$

$$= \frac{-12 \pm \sqrt{144 - 160}}{10}$$

$$= \frac{-12 \pm \sqrt{-16}}{10}$$

$$= \frac{-12 \pm 4i}{10}$$

$$\boxed{x = \frac{-6}{5} \pm \frac{2i}{5}}$$

4. $2w^2 + 4w = -3$

$+3 \quad +3$

$2w^2 + 4w + 3 = 0$

$a = 2$, $b = 4$, $c = 3$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(3)}}{2(2)}$$

$$= \frac{-4 \pm \sqrt{16 - 24}}{4}$$

$$= \frac{-4 \pm \sqrt{-8}}{4}$$

$$= \frac{-4 \pm 2i\sqrt{2}}{4}$$

$$\boxed{x = -1 \pm \frac{1}{2}i\sqrt{2}}$$

5. $2x(x + 1) = 7$

$2x^2 + 2x = 7$

$-7 \quad -7$

$2x^2 + 2x - 7 = 0$

$a = 2$, $b = 2$, $c = -7$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(2)(-7)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{4 + 56}}{4}$$

$$= \frac{-2 \pm \sqrt{60}}{4}$$

$\sqrt{60}$ / $4\sqrt{15}$ / (22)

$$= \frac{-2 \pm 2\sqrt{15}}{4}$$

$$= \boxed{\frac{-1}{2} \pm \frac{1}{2}\sqrt{15} \text{ or } \frac{-1 \pm \sqrt{15}}{2}}$$

6. $5 = 4r(2r + 3)$

$5 = 8r^2 + 12r$

$-5 \quad -5$

$0 = 8r^2 + 12r - 5$

$a = 8$

$b = 12$

$c = -5$

$$x = \frac{-12 \pm \sqrt{12^2 - 4(8)(-5)}}{2(8)}$$

$$= \frac{-12 \pm \sqrt{144 + 160}}{16}$$

$$= \frac{-12 \pm \sqrt{304}}{16}$$

$\sqrt{304}$: $304 = 16 \cdot 19$, $16 = 4 \cdot 4$

$$= \frac{-12 \pm 4\sqrt{19}}{16}$$

$$x = \frac{-3 \pm \sqrt{19}}{4}$$

7. $(3n - 5)(2n - 2) = 6$

$6n^2 - 6n - 10n + 10 = 6$

$-6 \quad -6$

$6n^2 - 16n + 4 = 0$

$a = 6$

$b = -16$

$c = 4$

$$x = \frac{16 \pm \sqrt{(-16)^2 - 4(6)(4)}}{2(6)}$$

$$= \frac{16 \pm \sqrt{256 - 96}}{12}$$

$$= \frac{16 \pm \sqrt{160}}{12}$$

$\sqrt{160}$: $160 = 16 \cdot 10$, $16 = 4 \cdot 4$

$$= \frac{16 \pm 4\sqrt{10}}{12}$$

$$x = \frac{4 \pm \sqrt{10}}{3}$$

8. $(2x + 1)(2x - 1) = 4x$

$4x^2 - 2x + 2x - 1 = 4x$

$-4x \quad -4x$

$4x^2 - 4x - 1 = 0$

$a = 4$

$b = -4$

$c = -1$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(4)(-1)}}{2(4)}$$

$$= \frac{4 \pm \sqrt{16 + 16}}{8}$$

$$= \frac{4 \pm \sqrt{32}}{8}$$

$\sqrt{32}$: $32 = 16 \cdot 2$, $16 = 4 \cdot 4$

$$= \frac{4 \pm 4\sqrt{2}}{8}$$

$$= \frac{1 \pm \sqrt{2}}{2}$$