Finding Quadratic Equations from Points

Earlier in the year we learned how to find an equation for a line in the form $y = mx + b$ if we knew the coordinates for two points on the line. In this section we will learn how to find the equation for a parabola in the form $y = ax^2 + bx + c$ if we know the coordinates for three points on the line.

Example: Find the equation for the parabola that passes through the points $(-1, 6), (-2, 17), and\ (2, 9)$.

Method 1. $ax^2 + bx + c = y$

$(-1, 6)$ $a(-1)^2 + b(-1) + c = 6$

[1] $a - b + c = 6$

$(-2, 17)$ $a(-2)^2 + b(-2) + c = 17$

[2] $4a - 2b + c = 17$

$(2, 9)$ $a(2)^2 + b(2) + c = 9$

[3] $4a + 2b + c = 9$

#2 + #3

$4a - 2b + c = 17$
$+\ 4a + 2b + c = 9$

#4 $8a + 2c = 26$

2(#1) + #3

$2a - 2b + 2c = 12$
$+4a + 2b + c = 9$

#5 $6a + 3c = 21$

3(#4) + (-2)(#5)

$24a + 6c = 78$
$-12a - 6c = -42$

$12a = 36$

$\frac{12a}{12} = \frac{36}{12}$

$a = 3$

#5

$6(3) + 3c = 21$
$-18 \qquad -18$

$3c = 3$

$\frac{3c}{3} = \frac{3}{3}$

$c = 1$

#1 $a = 3$ $c = 1$

$3 - b + 1 = 6$
$4 - b = 6$
$-4 \qquad -4$

$-b = 2$

$\frac{-b}{-1} = \frac{2}{-1}$

$b = -2$

$y = 3x^2 - 2x + 1$

Method 2 (Matrices): $ax^2 + bx + c = y$

$(-1, 6)$: $a - b + c = 6$

$(-2, 17)$: $4a - 2b + c = 17$

$(2, 9)$: $4a + 2b + c = 9$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 17 \\ 9 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$3x^2 - 2x + 1 = y$

Break for Practice:

1. Find the equation for the parabola that contains the points $(-2,-11),(4,13),$ *and* $(6,29)$.

$ax^2+bx+c=y$

$(-2,-11)$: $4a-2b+c=-11$

$(4,13)$: $16a+4b+c=13$

$(6,29)$: $36a+6b+c=29$

$[A^{-1}][B]$

$$\begin{bmatrix} 4 & -2 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \end{bmatrix} \begin{bmatrix} -11 \\ 13 \\ 29 \end{bmatrix} = \begin{bmatrix} .5 \\ 3 \\ -7 \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$$

$.5x^2+3x-7=y$

Extended Practice: Find the equation for the parabola that contains the points:

1. $(1,6),(3,26),(-2,21)$

$(1,6)$: $1a+1b+1c=6$

$(3,26)$: $9a+3b+c=26$

$(-2,21)$: $4a-2b+c=21$

$(A^{-1})(B)$

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 26 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$$

$y=3x^2-2x+5$

2. $(-2,-41),(-3,-72),(5,-48)$

$(-2,-41)$: $4a-2b+c=-41$

$(-3,-72)$: $9a-3b+c=-72$

$(5,-48)$: $25a+5b+c=-48$

$(A^{-1})(B)$

$$\begin{bmatrix} 4 & -2 & 1 \\ 9 & -3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} -41 \\ -72 \\ -48 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \\ -3 \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$$

$y=-4x^2+11x-3$

3. $(10,40),(-20,160),(-5,10)$

$(10,40)$: $100a+10b+c=40$

$(-20,160)$: $400a-20b+c=160$

$(-5,10)$: $25a-5b+c=10$

$(A^{-1})(B)$

$$\begin{bmatrix} 100 & 10 & 1 \\ 400 & -20 & 1 \\ 25 & -5 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 160 \\ 10 \end{bmatrix} = \begin{bmatrix} .4 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$$

$y=.4x^2$

4. $(-4,-37), (2,11), (0,-1)$

$(-4,-37)$: $16a - 4b + c = -37$

$(2,11)$: $4a + 2b + c = 11$

$(0,-1)$: $0a + 0b + c = -1$

$(A^{-1})(B)$

$$\begin{bmatrix} 16 & -4 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -37 \\ 11 \\ -1 \end{bmatrix} = \begin{bmatrix} -.5 \\ 7 \\ -1 \end{bmatrix}$$

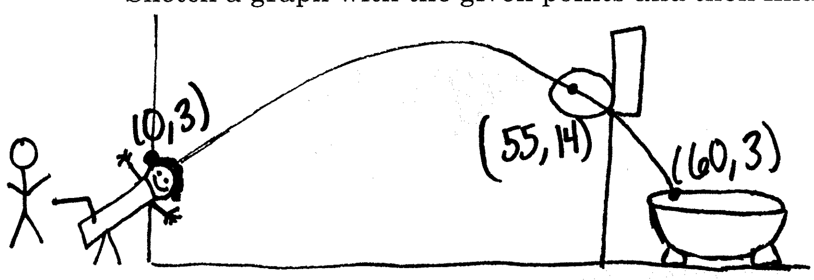
$$\boxed{y = -.5x^2 + 7x - 1}$$

Quadratic Applications

For the ½ time entertainment at a basketball game, a student agrees to be shot out of a cannon through an enlarged basketball hoop and into a tub of water. The person in the cannon is 3 feet off of the ground. The basketball hoop is 55 feet away and 14 feet off of the ground. The surface of the tub of water is 60 feet away and 3 feet off of the ground.

a) Another student, being a curious young mathematician, quickly finds the equation for this function. What kind of function is this? quadratic
Sketch a graph with the given points and then find the equation.

$ax^2 + bx + c = 0$



$(0,3)$ $0a + 0b + c = 3$

$(55,14)$ $3025a + 55b + c = 14$

$(60,3)$ $3600a + 60b + c = 3$

$(A^{-1})(B)$

$$\begin{bmatrix} 0 & 0 & 1 \\ 3025 & 55 & 1 \\ 3600 & 60 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 14 \\ 3 \end{bmatrix} = \begin{bmatrix} -.04 \\ 2.4 \\ 3 \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$$

$$\boxed{y = -.04x^2 + 2.4x + 3}$$

b) Four feet in front of the tub of water, the team lines up. One player is six feet six inches tall. Is he in danger of being hit?

$60 - 4 = 56$ $\quad y = -.04(56)^2 + 2.4(56) + 3$

$x = 56$ $\quad y \approx 11.96 \text{ ft}$

The flying student is 11.96 ft above ground at that point so the player is safe!

c) The roof is at 35 feet. Is the student being shot from the cannon safe?

Method 1 (Symmetry)

(0,3) and (60,3). The axis of symmetry is in the middle at x = 30.

$y = -.04(30)^2 + 2.4(30) + 3$

$y = 39$ The roof is too low!

Method 2 (Find the vertex)

$\left(\frac{-60}{2}\right)^2$ $(-30)^2$ 900

$y = -.04x^2 + 2.4x + 3$

$-3 \qquad -3$

$y - 3 = -.04(x^2 - 60 + 900)$

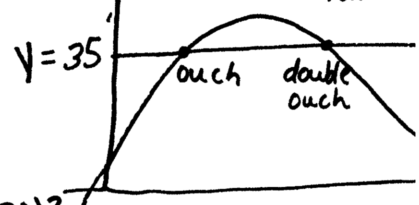
$y - 3 - 36 = -.04(x - 30)^2$

$y - 39 = -.04(x - 30)^2$

(30,39)

Method 3 (graph)

$Y_1 = 35$ $Y_2 = -.04x^2 + 2.4x$ Student



Extended Practice:

1. Assume that the number of liters of water remaining in the bathtub varies quadratically with the number of minutes, which have elapsed since you pulled the plug.

a) If the tub has 38.4, 21.6, and 9.6 liters remaining at 1, 2, and 3 minutes respectively, since you pulled the plug, write an equation expressing liters in terms of time.

(mins, liters)

(1, 38.4) $a + b + c = 38.4$

(2, 21.6) $4a + 2b + c = 21.6$

(3, 9.6) $9a + 3b + c = 9.6$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 38.4 \\ 21.6 \\ 9.6 \end{bmatrix}$$

$$A^{-1}(B) = \begin{bmatrix} 2.4 \\ -24 \\ 60 \end{bmatrix}$$

$y = 2.4x^2 - 24x + 60$

b) How much water was in the tub when you pulled the plug?

time $x = 0$ start

$y = 2.4(0)^2 - 24(0) + 60$

$y = 60$ 60 liters

c) When will the tub be empty.

$y = 0$

$0 = 2.4x^2 - 24x + 60$

$a = 2.4$, $b = -24$, $c = 60$

$$x = \frac{24 \pm \sqrt{(-24)^2 - 4(2.4)(60)}}{2(2.4)} = \frac{24 \pm \sqrt{0}}{4.8}$$

$x = 5$ mins

d) Why is a quadratic function more reasonable for this problem than a linear function would be?

The water does not drain at a constant rate. There is more pressure at the beginning so it drains faster than near the end when its almost empty.

2. Suppose that you are an actuary for an insurance agency. Your company plans to offer a senior citizen's accident policy, and you must predict the likelihood of an accident as a function of the driver's age. From previous accident records, you find the following information:

Age	Accidents per 100 million kilometers driven
20	440
30	280
40	200

You know that the number of accidents per 100 million kilometers driven should reach a minimum then go up again for very old drivers. Therefore, you assume that a quadratic function is reasonable.

a) Write the particular equation expressing accidents per 100 million kilometers in terms of age.

(20, 440): $400a + 20b + c = 440$

(30, 280): $900a + 30b + c = 280$

(40, 200): $1600a + 40b + c = 200$

$$A = \begin{bmatrix} 400 & 20 & 1 \\ 900 & 30 & 1 \\ 1600 & 40 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 440 \\ 280 \\ 200 \end{bmatrix}$$

$$A^{-1}(B) = \begin{bmatrix} .4 \\ -36 \\ 1000 \end{bmatrix}$$

$\boxed{y = .4x^2 - 36x + 1000}$

b) How many accidents per 100 million kilometers would you expect for an 80-year-old driver?

age, $x = 80$ $y = .4(80)^2 - 36(80) + 1000$

$\boxed{y = 680 \text{ accidents per 100 million km}}$

c) Based on your model, who is safer; a 16-year-old driver or a 70-year-old driver?

age, $x = 16$ $y = .4(16)^2 - 36(16) + 1000 = 526.4$ accidents

$x = 70$ $y = .4(70)^2 - 36(70) + 1000 = 440$ accidents

$\boxed{\text{70 yr old driver}}$

d) What age driver appears to be the safest?

Note the symmetry with 20 yr old and 70 yrs old. The safest should be the average (vertex)

$\frac{20+70}{2} = \boxed{\text{45 yr olds}}$

e) Your company decides to insure licensed drivers up to the age where the accident rate reaches 830 per 100 million kilometers. What is the domain of this quadratic function?

let $y = 830$

$.4x^2 - 36x + 1000 = 830$
$\quad\quad\quad\quad\quad -830 \;\; -830$

$.4x^2 - 36x + 170 = 0$
a b c

$x = \frac{36 \pm \sqrt{(-36)^2 - 4(.4)(170)}}{2(.4)}$

$x = \frac{36 \pm 32}{.8}$ → $\frac{36+32}{.8} = 85$; $\frac{36-32}{.8} = 5$ Not legal

$x = 5, 85$

$\boxed{\text{16 – 85 yr olds}}$