The Rational Root Theorem

In this section we will use an extension of the factor theorem, the graphing calculator/computer, and synthetic substitution to factor higher degree polynomials that have all rational roots.

Rational Root Theorem: $(ax - b)$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

Note: The maximum number of roots is equal to the degree of the polynomial.

Break for Practice:

1. Factor $\overset{a}{2x^3} - 3x^2 - 8x - \overset{b}{3}$

a) How many factors should we expect? 3

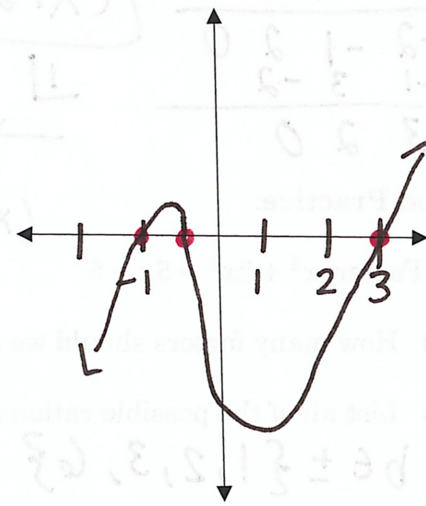
b) List all of the possible rational roots. $\frac{b}{a}$

$b \in \{\pm 1, \pm 3\}$, $a \in \{\pm 1, \pm 2\}$

$\frac{b}{a} \in \pm\{1, \frac{1}{2}, 3, \frac{3}{2}\}$

c) Sketch a graph.

rational roots are the x-intercepts $(y=0)$



d) Write the list of factors. $x = -1, 3, \underset{?}{\_\_}$

$$\begin{array}{r|rrrr} -1 & \overset{x^3}{2} & -3 & -8 & -3 \\ & \downarrow & -2 & 5 & 3 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

$2x^2 - 5x - 3$

$$\begin{array}{r|rrr} 3 & 2 & -5 & -3 \\ & \downarrow & 6 & 3 \\ \hline & 2 & 1 & 0 \end{array}$$

$2x+1$

* may use quadratic form. or factor if needed

$2x + 1 = 0$

$-1 \quad -1$

$\frac{2x}{2} = \frac{-1}{2}$

$x = -\frac{1}{2}$

List of factors (3)

$\boxed{(2x+1)(x+1)(x-3)}$

roots / solutions $= -1, 3, -\frac{1}{2}$

* signs opp.

$a=1$ $b=4$

2. Factor $x^4 - 5x^2 + 4$

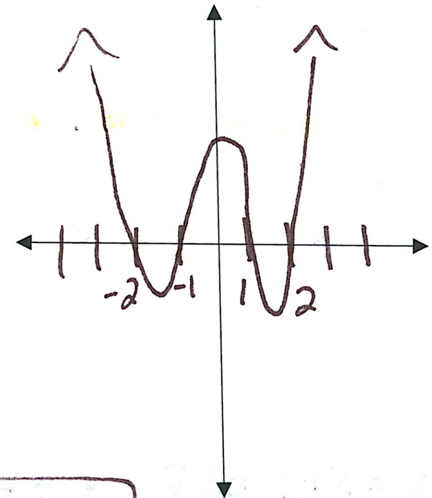
e) How many factors should we expect? 4

f) List all of the possible rational roots.

$b \in \pm\{1, 2, 4\}$ $a \in \pm\{1\}$

$\frac{b}{a} \in \pm\{1, 2, 4\}$

g) Sketch a graph.



*have to check

h) Write the list of factors. $-2, -1, 1, 2$

```
-2 | 1  0  -5  0  4
   | ↓ -2   4  2 -4
-1 | 1 -2  -1  2  0
   | ↓ -1   3 -2
     1 -3   2  0
```

$\boxed{(x+2)(x+1)(x-1)(x-2)}$

```
1 | 1 -3  2  0
  | ↓  1 -2
    1 -2  0
```

$(x-2)$

Extended Practice:

1. Factor $x^3 + 2x^2 - 5x - 6$

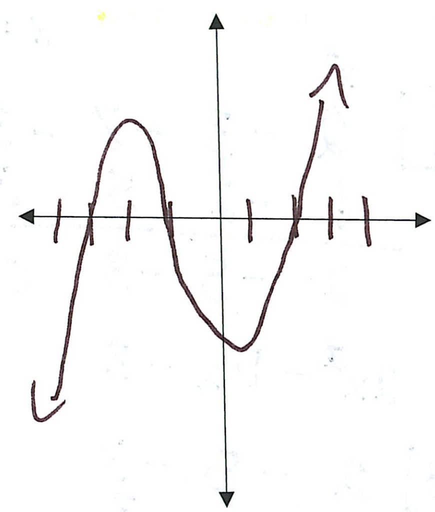
a) How many factors should we expect? 3

b) List all of the possible rational roots.

$b \in \pm\{1, 2, 3, 6\}$ $a \in \pm\{1\}$

$\frac{b}{a} \in \pm\{1, 2, 3, 6\}$

c) Sketch a graph.



d) Write the list of factors. $x = -3, -1, 2$

```
-3 | 1  2  -5  -6
   | ↓ -3   3   6
-1 | 1 -1  -2   0
   | ↓ -1   2
     1 -2   0
```

$\boxed{(x+3)(x+1)(x-2)}$

$(x-2)$

2. Factor $x^3 - 19x + 30$

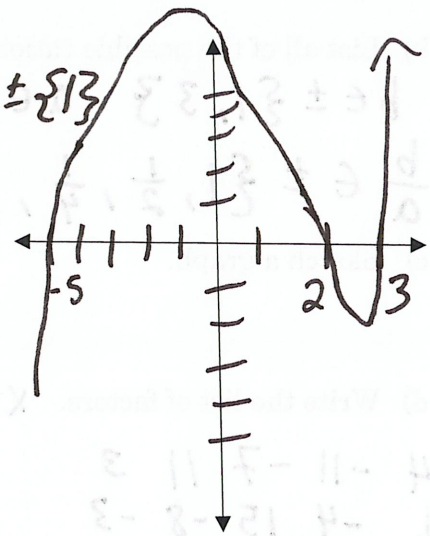
a) How many factors should we expect? $\underline{\quad 3 \quad}$

b) List all of the possible rational roots.

$b \in \pm\{1, 2, 3, 5, 6, 10, 15, 30\}$ $a \in \pm\{1\}$

$\frac{b}{a} \in \pm\{1, 2, 3, 5, 6, 10, 15, 30\}$

c) Sketch a graph.



d) Write the list of factors. $-5, 2, 3$

$$\begin{array}{r|rrrr} -5 & 1 & 0 & -19 & 30 \\ & \downarrow & -5 & 25 & -30 \\ \hline 2 & 1 & -5 & 6 & 0 \\ & \downarrow & 2 & -6 & \\ \hline & 1 & -3 & 0 & \end{array}$$

$(x-3)$

$\boxed{(x+5)(x-2)(x-3)}$

3. Factor $2x^3 + 5x^2 - 4x - 3$

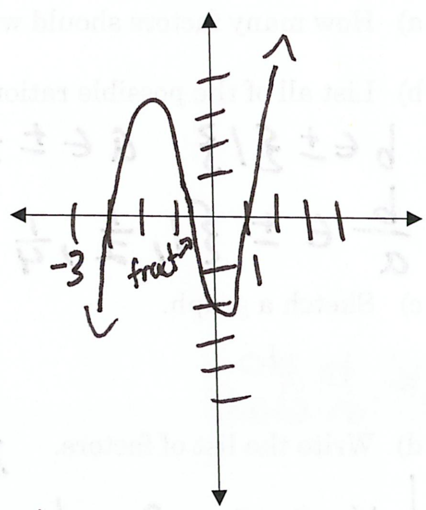
a) How many factors should we expect? $\underline{\quad 3 \quad}$

b) List all of the possible rational roots.

$b \in \pm\{1, 3\}$ $a \in \pm\{1, 2\}$

$\frac{b}{a} \in \pm\{1, \frac{1}{2}, 3, \frac{3}{2}\}$

c) Sketch a graph.



d) Write the list of factors. roots $\cancel{x=}-3, 1$ ___ ? fraction

$$\begin{array}{lr|rrrr} x^3 & -3 & 2 & 5 & -4 & -3 \\ & & \downarrow & -6 & 3 & 3 \\ \hline x^2 & 1 & 2 & -1 & -1 & 0 \\ & & \downarrow & 2 & 1 & \\ \hline x & & 2 & 1 & 0 & \end{array}$$

$\longrightarrow 2x+1$

$(2x+1) = 0$

$\quad -1 \quad -1$

$\frac{2x}{2} = \frac{-1}{2}$ $\quad$ root $x = -\frac{1}{2}$

factors

$\boxed{(2x+1)(x+3)(x-1)}$

4. Factor $4x^4 - 11x^3 - 7x^2 + 11x + 3$

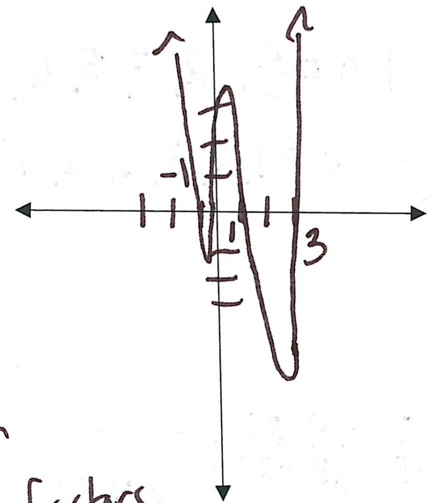
a) How many factors should we expect? $\underline{\quad 4 \quad}$

b) List all of the possible rational roots.

$b \in \pm \{1, 3\} \quad a \in \pm \{1, 2, 4\}$

$\frac{b}{a} \in \pm \{1, \frac{1}{2}, \frac{1}{4}, 3, \frac{3}{2}, \frac{3}{4}\}$

c) Sketch a graph.



roots

d) Write the list of factors. $x = -1, 1, 3$ ____

fraction ?

$$
\begin{array}{r|rrrrr}
x^4 \quad -1 & 4 & -11 & -7 & 11 & 3 \\
 & \downarrow & -4 & 15 & -8 & -3 \\
\hline
x^3 \quad 1 & 4 & -15 & 8 & 3 & 0 \\
 & \downarrow & 4 & -11 & -3 & \\
\hline
x^2 \quad 3 & 4 & -11 & -3 & 0 & \\
 & \downarrow & 12 & 3 & & \\
\hline
x \quad & 4 & 1 & 0 & &
\end{array}
$$

$\boxed{4x+1}$

$4x + 1 = 0$

$\quad -1 \quad -1$

$\frac{4x}{4} = \frac{-1}{4}$

root

$x = -\frac{1}{4}$

factors

$\boxed{(x+1)(x-1)(x-3)(4x+1)}$

5. Factor $4x^4 + 8x^3 + 3x^2 - 2x - 1$

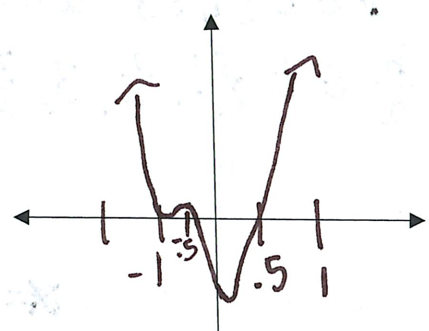
a) How many factors should we expect? $\underline{\quad 4 \quad}$

b) List all of the possible rational roots.

$b \in \pm \{1\} \quad a \in \pm \{1, 2, 4\}$

$\frac{b}{a} \in \pm \{1, \frac{1}{2}, \frac{1}{4}\}$

c) Sketch a graph.



* Ask to do in class

d) Write the list of factors. $x = -1, \frac{1}{2}$

$$
\begin{array}{r|rrrrr}
x^4 \quad -1 & 4 & 8 & 3 & -2 & -1 \\
 & \downarrow & -4 & -4 & 1 & 1 \\
\hline
x^3 \quad \frac{1}{2} & 4 & 4 & -1 & -1 & 0 \\
 & \downarrow & 2 & 3 & 1 & \\
\hline
x^2 \quad & 4 & 6 & 2 & 0 &
\end{array}
$$

$4x^2 + 6x + 2 = 0$

$2(2x^2 + 3x + 1) = 0$

$2(2x+1)(x+1) = 0$

$2x + 1 = 0$

$\quad -1 \quad -1$

$\underline{2x} = \underline{-1}$

$x = -\frac{1}{2}$

$x + 1 = 0$

$\underline{\quad -1 \quad -1}$

$x = -1$

* 2 double roots (same)

factors

$\boxed{(2x+1)(x+1)(x+1)(2x-1)}$

6. Factor $x^5 + 2x^4 - 10x^3 - 20x^2 + 9x + 18$

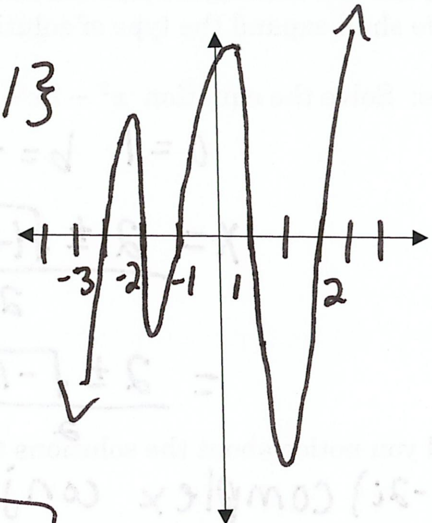
a) How many factors should we expect? 5

b) List all of the possible rational roots.

$b \in \pm \{1, 2, 3, 6, 9, 18\}$ $a \in \pm \{1\}$

$\frac{b}{a} \in \pm \{1, 2, 3, 6, 9, 18\}$

c) Sketch a graph.



d) Write the list of factors.

$x = -3, -2, -1, 1, 3$

$\boxed{(x+3)(x+2)(x+1)(x-1)(x-3)}$

-3 | 1 2 -10 -20 9 18
↓ -3 3 21 -3 -18
-2 | 1 -1 -7 1 6 0
↓ -2 6 2 -6
-1 | 1 -3 -1 3 0
↓ -1 4 -3
1 | 1 -4 3 0
↓ 1 -3
1 -3 0
x-3 ✓

7. Factor $x^5 + 15x^4 + 85x^3 + 225x^2 + 274x + 120$

a) How many factors should we expect? 5

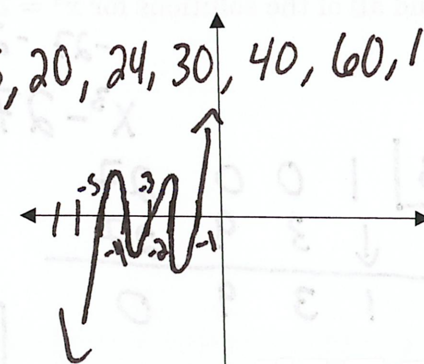
b) List all of the possible rational roots.

$b \in \pm \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$

$a \in \pm \{1\}$

$\frac{b}{a} \in \pm \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$

c) Sketch a graph.



d) Write the list of factors.

$x = -5, -4, -3, -2, -1$

$\boxed{(x+5)(x+4)(x+3)(x+2)(x+1)}$

-5 | 1 15 85 225 274 120
↓ -5 -50 -175 -250 -120
-4 | 1 10 35 50 24 0
↓ -4 -24 -44 -24
-3 | 1 6 11 6 0
↓ -3 -9 -6
-2 | 1 3 2 0
↓ -2 -2
1 1 0
x+1 ✓

Some Useful Theorems

In the last section, we learned how to find all <u>rational</u> solutions to polynomial equations. In this section we shall expand the type of solutions that we are able to find.

Example: Solve the equation $x^2 - 2x + 5 = 0$

$a=1 \quad b=-2 \quad c=5$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

What did you notice about the solutions to the above equation?

$(1+2i)(1-2i)$ complex conjugates (middle sign different)

Theorem: If a polynomial has <u>real</u> coefficients, then if there is a <u>complex</u> solution, there will actually be a <u>pair</u> of <u>complex</u> solutions, and they will be <u>complex</u> <u>conjugates</u>. $(a+bi)(a-bi)$

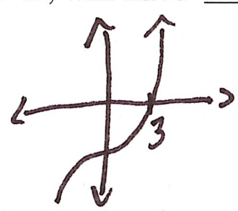
There is another important theorem that states:

Theorem: A <u>polynomial</u> equation of degree n , will have <u>n</u> solutions. (These include real, imaginary, complex, and duplicates.)

Break for Practice: graph first for real zeros

1. Find all of the solutions for $x^3 = 27$.

3 solutions



$-27 \quad -27$

$x^3 - 27 = 0 \qquad x = 3$ obv. answer

$$\begin{array}{r|rrrr} 3 & 1 & 0 & 0 & -27 \\ & \downarrow & 3 & 9 & 27 \\ \hline & 1 & 3 & 9 & 0 \end{array}$$

$1x^2 + 3x + 9 = 0$

a b c

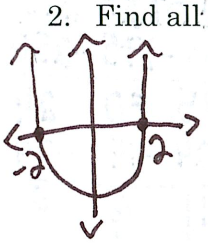
$\sqrt{9-36} = \sqrt{-27}$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{-3 \pm 3i\sqrt{3}}{2}$$

$$\boxed{x = 3, \frac{-3}{2} \pm \frac{3i\sqrt{3}}{2}}$$

2. Find all of the solutions for $x^4 = 16$.



$-16 \quad -16$

$x^4 - 16 = 0$

$x = 2, -2$

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & 0 & 0 & -16 \\ & \downarrow & 2 & 4 & 8 & 16 \\ \hline & 1 & 2 & 4 & 8 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 4 & 8 \\ & \downarrow & -2 & 0 & -8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$x^2 + 4 = 0$

$\rightarrow \sqrt{x^2} = \sqrt{-4}$

$x = \pm 2i$

$$\boxed{x = 2, -2, 2i, -2i}$$

3. Find a second degree equation with solutions of x = 3, and x = -4.

$0=(x-3)(x+4)$

$x^2+4x-3x-12$

$\boxed{x^2+x-12=0}$

4. Find a third degree equation with solutions x = 3*i*, and x = 5.

$(x+3i)\,(x-3i)\,(x-5)$

$(x^2-\cancel{3xi}+\cancel{3xi}-9i^2)(x-5)$

$(x^2+9)(x-5)=0$

$\boxed{x^3-5x^2+9x-45=0}$

Extended Practice:

1. Solve the following equations.

a) $x^2+3x+5=0$

$a=1 \quad b=3 \quad c=5$

$x=\dfrac{-3\pm\sqrt{3^2-4(1)(5)}}{2(1)}$

$=\dfrac{-3\pm\sqrt{-11}}{2}$

$\boxed{x=\dfrac{-3}{2}\pm\dfrac{i\sqrt{11}}{2}}$

b) $x^2-3x+6=0$

$a=1 \quad b=-3 \quad c=6$

$x=\dfrac{3\pm\sqrt{(-3)^2-4(1)(6)}}{2(1)}$

$=\dfrac{3\pm\sqrt{-15}}{2}$

$\boxed{x=\dfrac{3}{2}\pm\dfrac{i\sqrt{15}}{2}}$

c) $-x^2-2x-6=0$

$a=-1 \quad b=-2 \quad c=-6$

$x=\dfrac{2\pm\sqrt{(-2)^2-4(-1)(-6)}}{2(-1)}$

$=\dfrac{2\pm\sqrt{-20}}{-2}$ $\quad i\sqrt{20}$ $\quad 4 \; 5$ $\quad 2\,2$

$=\dfrac{2\pm 2i\sqrt{5}}{-2}$

$\boxed{x=-1\mp i\sqrt{5}}$

2. Find all of the solutions for the following equations.

a) $x^3=8$

$-8 \quad -8$

$x^3-8=0$

$c=2$

$x=2$

$2\,|\;1 \quad 0 \quad 0 \quad -8$
$\;\;\downarrow \quad 2 \quad 4 \quad 8$
$\;\;1 \quad 2 \quad 4 \quad 0$

$x^2+2x+4=0$

$a=1 \quad b=2 \quad c=4$

$x=\dfrac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}$

$=\dfrac{-2\pm\sqrt{-12}}{2}$

$=\dfrac{-2\pm 2i\sqrt{3}}{2}$

$\boxed{x=-1\pm i\sqrt{3},\,2}$

b) $x^3=125$

$-125 \quad -125$

$x=5$

$x^3-125=0$

$5\,|\;1 \quad 0 \quad 0 \quad -125$
$\;\;\downarrow \quad 5 \quad 25 \quad 125$
$\;\;1 \quad 5 \quad 25 \quad 0$

$x^2+5x+25=0$

$a=1 \quad b=5 \quad c=25$

$x=\dfrac{-5\pm\sqrt{5^2-4(1)(25)}}{2(1)}$

$=\dfrac{-5\pm\sqrt{-75}}{2}$

$x=\dfrac{-5\pm 5i\sqrt{3}}{2}$

$\boxed{x=\dfrac{-5}{2}\pm\dfrac{5i\sqrt{3}}{2},\,5}$

a) $x^4 = 16$ $X = 2, -2$

$-16 \quad -16$

$x^4 - 16 = 0$

$\underline{2}| \; 1 \; 0 \; 0 \; 0 \; -16$

$\downarrow \; 2 \; 4 \; 8 \; 16$

$\underline{-2}| \; 1 \; 2 \; 4 \; 8 \; 0 \quad x^3$

$\downarrow \; -2 \; 0 \; -8$

$1 \; 0 \; 4 \; 0 \quad x^2$

$x^2 + 4 = 0$

$-4 \quad -4$

$\sqrt{x^2} = \sqrt{-4}$

$x = \pm 2i$

$\boxed{x = 2, -2, \pm 2i}$

b) $x^4 = 81$ $X = 3, -3$

$-81 \quad -81$

$x^4 - 81 = 0$

$\underline{3}| \; 1 \; 0 \; 0 \; 0 \; -81$

$\downarrow \; 3 \; 9 \; 27 \; 81$

$\underline{-3}| \; 1 \; 3 \; 9 \; 27 \; 0$

$\downarrow \; -3 \; 0 \; -27$

$1 \; 0 \; 9 \; 0$

$x^2 + 9 = 0$

$-9 \quad -9$

$\sqrt{x^2} = \sqrt{-9}$

$x = \pm 3i$

$\boxed{x = 3, -3, \pm 3i}$

3. Write a second-degree equation which has solutions of x = 5, and x = -2.

$0 = (x-5)(x+2)$

$0 = x^2 + 2x - 5x - 10$

$\boxed{0 = x^2 - 3x - 10}$

4. Write a third degree equation which includes solutions of x = 4*i*, and x = 5.

$0 = (x+4i)(x-4i)(x-5)$

$= (x^2 - 4xi + 4xi - 16i^2)(x-5)$

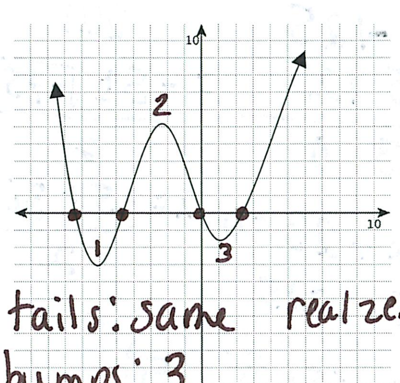
$= (x^2 + 16)(x-5)$

$\boxed{0 = x^3 - 5x^2 + 16x - 80}$

Graphing Higher Degree Polynomial Functions

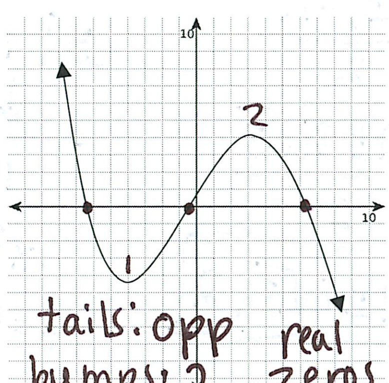
In this section we will be working with the graphs of higher degree polynomial functions. You will be looking for patterns in the shapes of the graphs. You will also learn how to tell how many zeros are real, and how many are imaginary. First you need to learn a little background.

For each graph tell what you notice about the direction of the "tails", how many bumps (relative max and min) that you see, and the number of real zeros that are on the graph.



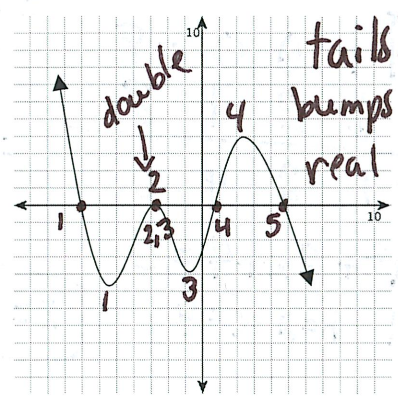
tails: same real zeros: 4

bumps: 3



tails: opp real zeros 3

bumps: 2



tails: opp

bumps: 4

real zeros: 5

Break for Practice: For each polynomial, draw the graph and write the answers to the following questions.

Questions: a) What is the direction of the tails?
b) What is the degree of the polynomial?
c) How many bumps (relative max and min) are in the graph?
d) How many real zeros are there?
e) How many imaginary zeros are there?

$y = 3x + 4$

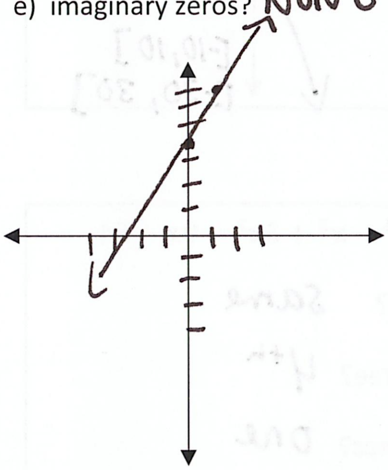
a) tails? opposite

b) degree? first

c) bumps? NONE

d) real zeros? one

e) imaginary zeros? NONE or 0



$y = -\frac{2}{3}x - 5$

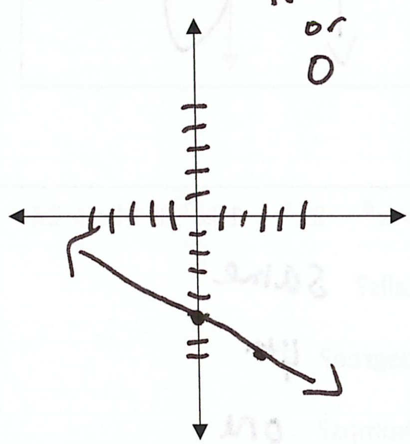
a) tails? opposite

b) degree? first

c) bumps? NONE

d) real zeros? one

e) imaginary zeros? NONE or 0



$y = 2x^2 + 5x - 3$

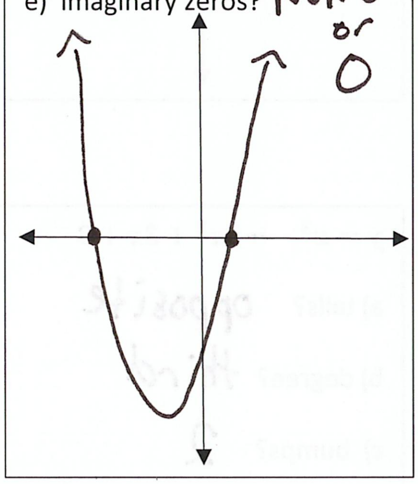
a) tails? same

b) degree? second

c) bumps? one

d) real zeros? two

e) imaginary zeros? NONE or 0



$y = -x^2 + 8x - 16$

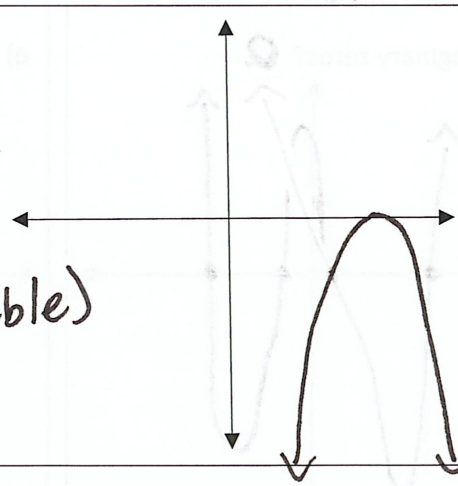
a) tails? same

b) degree? second

c) bumps? One

d) real zeros? two (double)

e) imaginary zeros? NONE



$y = x^2 + 3x + 5$

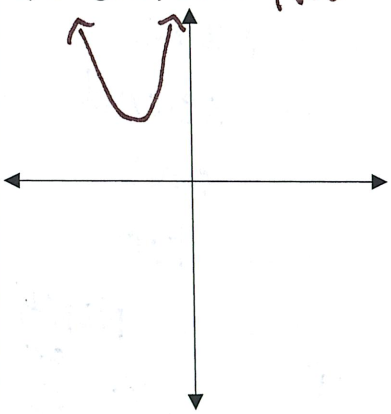
a) tails? same

b) degree? second

c) bumps? one

d) real zeros? NONE

e) imaginary zeros? two



$y = x^3 + 2x^2 - 5x - 6$

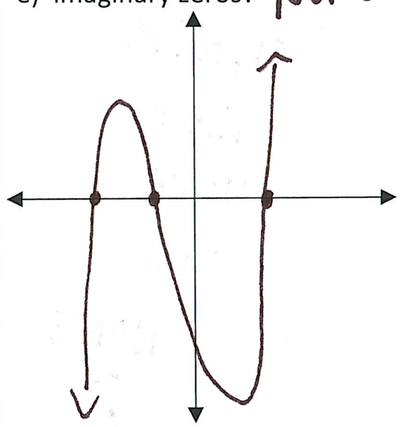
a) tails? opposite

b) degree? third

c) bumps? two

d) real zeros? 3

e) imaginary zeros? NONE



$y = x^3 - 3x^2 + 4x - 12$

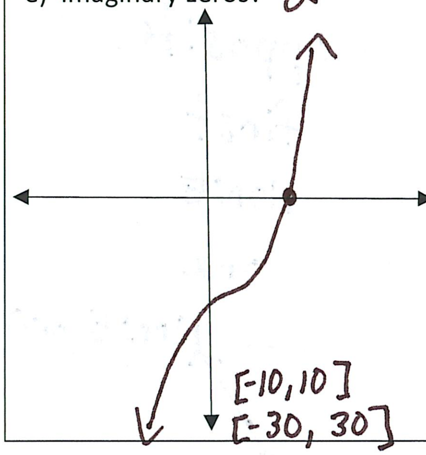
a) tails? opposite

b) degree? third

c) bumps? 0

d) real zeros? one

e) imaginary zeros? 2



$y = x^3 - 5x^2 + 3x + 9$

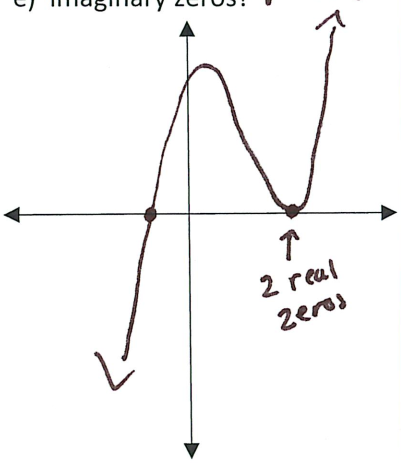
a) tails? opposite

b) degree? third

c) bumps? 2

d) real zeros? 3

e) imaginary zeros? NONE



$y = x^4 - 2x^3 - 13x^2 + 14x + 24$

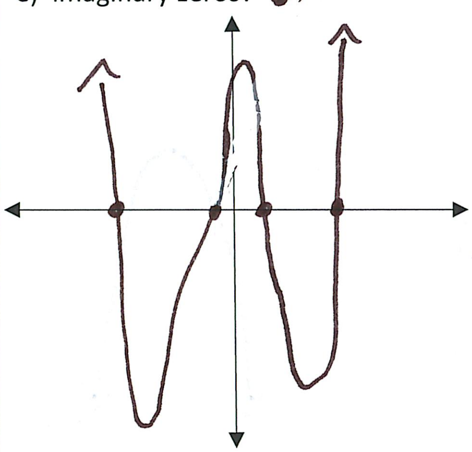
a) tails? same

b) degree? 4th

c) bumps? one

d) real zeros? 4

e) imaginary zeros? 0



X [-10, 10]

Y [-30, 30]

$y = x^4 - x^3 + 3x^2 - 9x - 54$

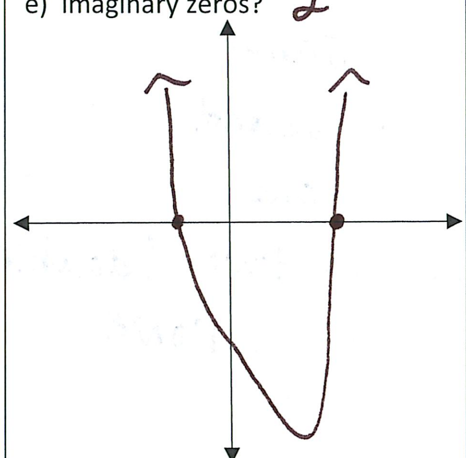
a) tails? same

b) degree? 4th

c) bumps? one

d) real zeros? 2

e) imaginary zeros? 2



X [-10, 10]

Y [-60, 60]

$y = x^5 + 3x^4 - 7x^3 - 11x^2 + 6x + 8$

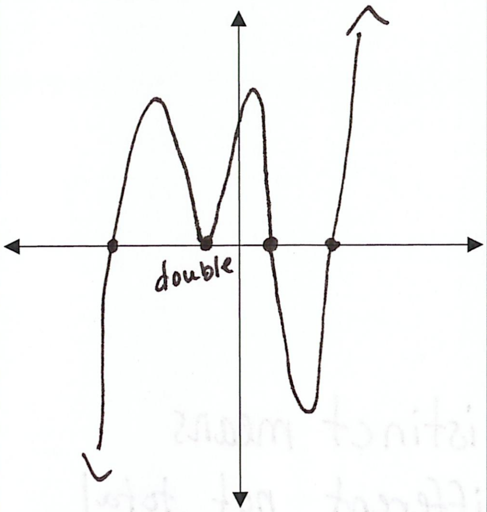
a) tails? Opposite

b) degree? 5th

c) bumps? 4

d) real zeros? 5

e) imaginary zeros? NONE



x [-10,10]
y [-40,90]

$y = x^5 - 2x^4 + 40x^2 - 41x - 78$

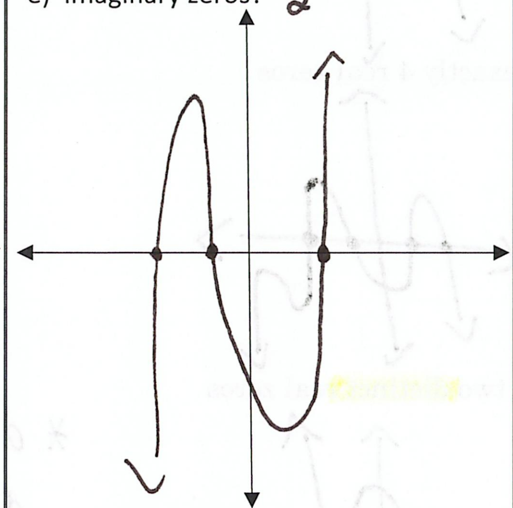
a) tails? opposite

b) degree? 5th

c) bumps? 2

d) real zeros? 3

e) imaginary zeros? 2



x [-10,10]
y [-100,100]

$y = x^6 - 14x^4 + 49x^2 - 36$

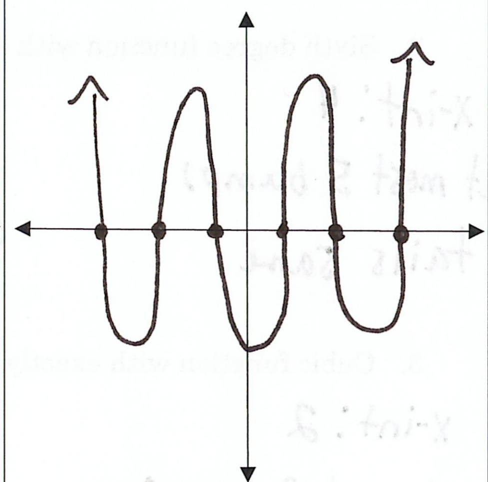
a) tails? same

b) degree? 6th

c) bumps? 5

d) real zeros? 6

e) imaginary zeros? NONE



[-10,10]
[-100,100]

$y = x^6 + 2x^5 - 3x^4 + 84x^3 - 184x^2 + 304x - 624$

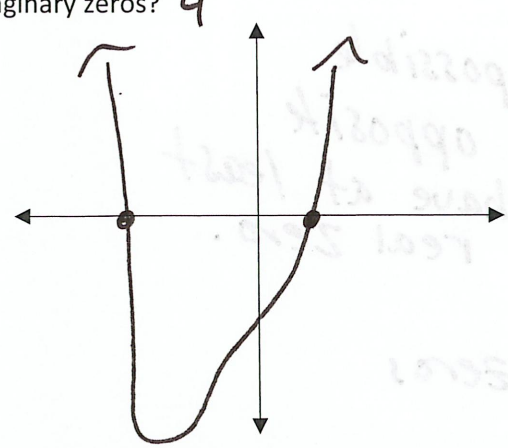
a) tails? same

b) degree? 6th

c) bumps? 1

d) real zeros? 2

e) imaginary zeros? 4



[-10,10]
[-10,000, 500]

Questions:

1. What is true about the direction of the tails in an odd degree function?

   opposite direction

2. What is true about the direction of the tails in an even degree function?

   same direction

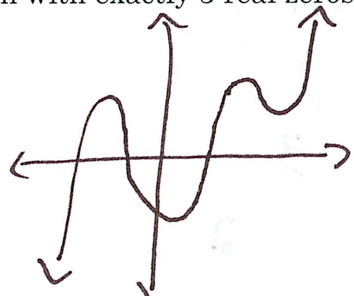
3. How does the maximum number of bumps (relative max and min) compare to the degree of a function?

   it is one less than the degree of the function

Extended Practice: Use what you have observed about the graphs of higher degree functions to sketch graphs of the functions described.

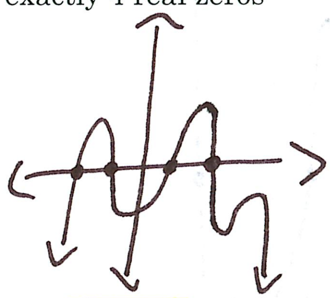
1. Quintic (5^{th} degree) function with exactly 3 real zeros

x-int: 3
at most 4 bumps
tails: opp



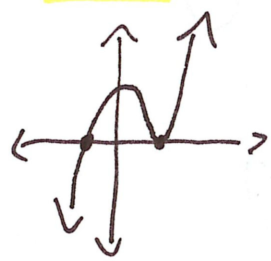
2. Sixth degree function with exactly 4 real zeros

x-int: 4
at most 5 bumps
tails same



3. Cubic function with exactly two distinct real zeros

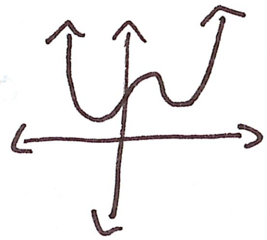
x-int: 2
at most 2 bumps
tails: opp



* distinct means different not total of 2

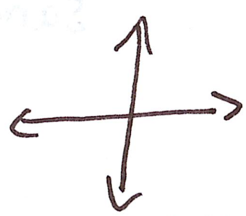
4. Quartic (4^{th} degree) function with no real zeros

x-int: NONE
at most 3 bumps
tails: same



5. Cubic function with no real zeros

x-int: NONE
at most 2 bumps
tails: opp



Not possible
tails opposite
will have at least
one real zero.

6. Quartic function with exactly five real zeros

x-int: 5
at most 3 bumps
tails: same

quartic = at most 4 zeros
Not possible