Algebra II
Unit 8
Polynomial Equations and Variations

Unit "I can" statements:

1. I can divide polynomials using long division.
2. I can divide polynomials using synthetic division.
3. I can use the remainder and factor theorems to find factors of polynomials and to solve polynomial equations.
4. I can apply the rational root theorem to find all of the rational roots of a polynomial function.
5. I can draw appropriate graphs for higher degree polynomial functions when given key facts about the functions.
6. I can find all rational and complex zeros of higher degree polynomial functions.
7. I can solve problems using direct, inverse, and joint variations.
8. I can use linear interpolation to find values not listed in a given table of data.

Common Core State Standards that are addressed in this unit include: A.CED.2a, A.CED.4a, A.APR.2b, A.APR.6d, N.CN.7c, N.CN.9c, F.IF.8
For more information see www.corestandards.org/Math/

Dividing Polynomials with Long Division

In this unit we will be exploring higher degree polynomial equations and functions. In order to solve these equations or graph these functions, it is often necessary to be able to identify the factors of the polynomial. One way to check if a number/expression is a factor of another is by using division. A factor will leave a remainder of zero.

In this unit we will learn two different ways to divide polynomials. Understanding the situation will help you to decide which method to apply.

The first method that we will spend time with is long division. The chief pro of the long division method is that it will work in all situations with all types of polynomials. The process is similar to normal long division with plain numbers.

Review: Divide 2560 by 12 using long division.

$213.\overline{3}$

$12\overline{)2560.0}$
-24
16
-12
40
-36
40
-36

$\boxed{213.\overline{3}}$

Now we will apply the same process to polynomials. **Note**: Before using long division, always write both polynomials in descending order, and insert any "missing terms" by using a coefficient of zero.

$0x^{\#}$

Break for Practice: Divide

1. $\frac{3x^3-2x^2-13x+10}{x-2}$

$3x^2+4x-5$

$x-2\overline{)3x^3-2x^2-13x+10}$
$+(-3x^3+6x^2)$
$4x^2-13x$
$+(-4x^2+8x)$
$-5x+10$
$+(+5x-10)$
0

$\boxed{3x^2+4x-5}$

2. $\frac{x^3-4x^2-19x+9}{x+3}$

$x^2-7x+2+\frac{3}{x+3}$

$x+3\overline{)x^3-4x^2-19x+9}$
$+(-x^3-3x^2)$
$-7x^2-19x$
$+(+7x^2+21x)$
$2x+9$
$-(-2x-6)$
3

$\boxed{x^2-7x+2+\frac{3}{x+3}}$

3. $\frac{x^3+8}{x+2}$

$$x^2-2x+4$$

$$x+2 \overline{)\, x^3+0x^2+0x+8}$$
$$+(-x^3-2x^2)$$
$$-2x^2+0x$$
$$-(+2x^2+4x)$$
$$4x+8$$
$$+(-4x-8)$$
$$0$$

4. $\frac{6x^3-19x^2+15}{3x-5}$

$$2x^2-3x-5-\frac{10}{3x-5}$$

$$3x-5 \overline{)\, 6x^3-19x^2+0x+15}$$
$$+(-6x^3+10x^2)$$
$$-9x^2+0x$$
$$-(+9x^2-15x)$$
$$-15x+15$$
$$+(+15x-25)$$
$$-10$$

5. $\frac{x^4-x^3+7x+5}{x^2+2x+1}$

$$x^2-3x+5$$

$$x^2+2x+1 \overline{)\, x^4-x^3+0x^2+7x+5}$$
$$+(-x^4-2x^3-x^2)$$
$$-3x^3-x^2+7x$$
$$+(+3x^3+6x^2+3x)$$
$$5x^2+10x+5$$
$$+(-5x^2-10x-5)$$
$$0$$

Extended Practice: Divide by using long division.

1. $\frac{x^2+3x-4}{x+2}$

$$x+1-\frac{6}{x+2}$$

$$x+2 \overline{)\, x^2+3x-4}$$
$$+(-x^2-2x)$$
$$x-4$$
$$+(-x-2)$$
$$-6$$

2. $\frac{x^2-x+3}{x+1}$

$$x-2+\frac{5}{x+1}$$

$$x+1 \overline{)\, x^2-x+3}$$
$$+(-x^2-x)$$
$$-2x+3$$
$$+(+2x+2)$$
$$5$$

3. $\frac{9z-z^2}{z-3}$

$$
\begin{array}{r}
-z+6+\frac{18}{z-3} \\
z-3\,\overline{)\,-z^2+9z+0} \\
+(+z^2-3z) \\
\hline
6z+0 \\
+(-6z+18) \\
\hline
18
\end{array}
$$

Answer: $-z+6+\frac{18}{z-3}$

4. $\frac{x^3-x^2-10x+10}{x-3}$

$$
\begin{array}{r}
x^2+2x-4-\frac{2}{x-3} \\
x-3\,\overline{)\,x^3-x^2-10x+10} \\
+(-x^3+3x^2) \\
\hline
2x^2-10x \\
+(-2x^2+6x) \\
\hline
-4x+10 \\
+(+4x-12) \\
\hline
-2
\end{array}
$$

Answer: $x^2+2x-4-\frac{2}{x-3}$

5. $\frac{4t^2-4t+1}{2t+1}$

$$
\begin{array}{r}
2t-3+\frac{4}{2t+1} \\
2t+1\,\overline{)\,4t^2-4t+1} \\
+(-4t^2-2t) \\
\hline
-6t+1 \\
+(+6t+3) \\
\hline
4
\end{array}
$$

Answer: $2t-3+\frac{4}{2t+1}$

6. $\frac{6u^2+7u+5}{3u-1}$

$$
\begin{array}{r}
2u+3+\frac{8}{3u-1} \\
3u-1\,\overline{)\,6u^2+7u+5} \\
+(-6u^2+2u) \\
\hline
9u+5 \\
+(-9u+3) \\
\hline
8
\end{array}
$$

Answer: $2u+3+\frac{8}{3u-1}$

7. $\frac{2s^3-29s+13}{s+4}$

$$
\begin{array}{r}
2s^2-8s+3+\frac{1}{s+4} \\
s+4\,\overline{)\,2s^3+0s^2-29s+13} \\
+(-2s^3-8s^2) \\
\hline
-8s^2-29s \\
+(+8s^2+32s) \\
\hline
3s+13 \\
+(-3s-12) \\
\hline
1
\end{array}
$$

Answer: $2s^2-8s+3+\frac{1}{s+4}$

8. $\frac{15z^3-z^2-11z-3}{3z^2-2z-1}$

$$
\begin{array}{r}
5z+3 \\
3z^2-2z-1\,\overline{)\,15z^3-z^2-11z-3} \\
+(-15z^3+10z^2+5z) \\
\hline
9z^2-6z-3 \\
+(-9z^2+6z+3) \\
\hline
0
\end{array}
$$

Answer: $5z+3$

Dividing Polynomials with Synthetic Division

In this section, we will learn how to divide polynomials using a technique called synthetic division. The pros of this method include its speed and compactness. The con is that it can only be used when dividing by polynomials in the form $x \pm b$. Remember to write all polynomials in descending order and insert any "missing terms."

Steps
1) identify "c" comes from divisor
* $(x - c) = 0$ (has to be "1x")
2) line up coefficients of all x^n terms (use 0 if missing)
3) bring down lead coefficient
4) multiply by "c" value then add to next coefficient
5) repeat step 4 until done
6) if remainder occurs write over divisor $(x-c)$
7) answer is coefficients dropped by a degree of 1 plus remainder (if one)

Example: Divide with synthetic division.

$$\frac{3x^3 - 2x^2 - 13x + 10}{x - 2}$$

c = 2

c	x^3	x^2	x	c
2	3	-2	-13	10
	↓	6	8	-10
	3	4	-5	0

*if 0 it means there is no remainder

$3x^2 + 4x - 5$

Break for Practice: Divide by using synthetic division

1. $\frac{x^2+3x-4}{x+2}$

c = -2

	x^2	x	c
-2	1	3	-4
	↓	-2	-2
	1	1	-6

$x + 1 - \frac{6}{x+2}$

2. $\frac{x^3+8}{x+2}$

c = -2

	x^3	x^2	x	c
-2	1	0	0	8
	↓	-2	4	-8
	1	-2	4	0

$x^2 - 2x + 4$

3. $\frac{6x^3-7x^2+3x+2}{3x+1}$

*factor den.

$3(x + \frac{1}{3})$

$x + \frac{1}{3} = 0$

$-\frac{1}{3} \quad -\frac{1}{3}$

$x = -\frac{1}{3} = c$

	x^3	x^2	x	c
$-\frac{1}{3}$	6	-7	3	2
	↓	-2	3	-2
	6	-9	6	0

$\frac{6x^2 - 9x + 6}{3} = 2x^2 - 3x + 2$

4. $\frac{6t^3+t^2+7t+10}{3t+2}$

$3\left(t+\frac{2}{3}\right)$

$c = -\frac{2}{3}$

	t^3	t^2	t	c
$-\frac{2}{3}$	6	1	7	10
	↓	−4	2	−6
	6	−3	9	4

$\frac{6t^2-3t+9}{3} + \frac{4}{3\left(t+\frac{2}{3}\right)} = \boxed{2t^2-t+3+\frac{4}{3t+2}}$

Extended Practice: Divide by using synthetic division.

1. $\frac{3x^3-5x^2+x-2}{x-2}$

	x^3	x^2	x	c
2	3	−5	1	−2
	↓	6	2	6
	3	1	3	4

$\boxed{3x^2+x+3+\frac{4}{x-2}}$

2. $\frac{x^3+3x^2-2x-6}{x+3}$

	x^3	x^2	x	c
−3	1	3	−2	−6
	↓	−3	0	6
	1	0	−2	0

$\boxed{x^2-2}$

3. $\frac{t^4+5t^3-2t-7}{t+5}$

	t^4	t^3	t^2	t	c
−5	1	5	0	−2	−7
	↓	−5	0	0	10
	1	0	0	−2	3

$\boxed{t^3-2+\frac{3}{t+5}}$

4. $\frac{2s^4-7s^3+7s+6}{s-3}$

	s^4	s^3	s^2	s	c
3	2	−7	0	7	6
	↓	6	−3	−9	−6
	2	−1	−3	−2	0

$\boxed{2s^3-s^2-3s-2}$

5. $\frac{x^5-1}{x-1}$

	x^5	x^4	x^3	x^2	x	c
1	1	0	0	0	0	-1
	↓	1	1	1	1	1
	1	1	1	1	1	0

$x^4+x^3+x^2+x+1$

6. $\frac{2x^4+x^3-x-2}{x+1}$

	x^4	x^3	x^2	x	c
-1	2	1	0	-1	-2
	↓	-2	1	-1	2
	2	-1	1	-2	0

$2x^3-x^2+x-2$

7. $\frac{2x^3-3x^2+4x-2}{2x+1}$

$2(x+\frac{1}{2})$

	x^3	x^2	x	c
$-\frac{1}{2}$	2	-3	4	-2
	↓	-1	2	-3
	2	-4	6	-5

$\frac{2x^2-4x+6}{2} - \frac{5}{2(x+\frac{1}{2})}$

$x^2-2x+3-\frac{5}{2x+1}$

8. $\frac{6t^4+5t^3-10t+4}{3t-2}$

$3(t-\frac{2}{3})$

	t^4	t^3	t^2	t	c
$\frac{2}{3}$	6	5	0	-10	4
	↓	4	6	4	-4
	6	9	6	-6	0

$\frac{6t^3+9t^2+6t-6}{3}$

$2t^3+3t^2+2t-2$

The Remainder and Factor Theorems

In this section we will look at two closely related theorems that will aid us when we begin factoring and graphing higher degree polynomial functions.

Consider: $P(x) = x^2 - 5x + 6$

Evaluate $P(1) = 1^2 - 5(1) + 6$

$= 1 - 5 + 6$

$P(1) = 2$

Now try this:

	x^2	x	c
1	1	-5	6
	↓	1	-4
	1	-4	2

Evaluate $P(2) = (2^2) - 5(2) + 6$

$= 4 - 10 + 6$

$= -6 + 6$

$P(2) = 0$

Now try this:

2	1	-5	6
	↓	2	-6
	1	-3	0

From these examples, we can see that the remainder in synthetic division can also be used to find the value of a function at x. If the remainder is zero, then the value of the function is zero, and a factor of the polynomial would be in the form (x – that value). These are the ideas stated in the Remainder and Factor Theorems.

Remainder Theorem: You can evaluate a polynomial at a certain value by just putting that value in the box of synthetic division. The __number__ in the __remainder__ position is the __value__ of the function.

Factor Theorem: __$(x - b)$__ is a factor of P(x) if and only if P(b) = __0__.

Break for Practice:

1. Use synthetic substitution for find P(c).

   a) $P(x) = x^3 - 3x^2 - 4x + 6 \quad c = 3$

3	1	-3	-4	6
	↓	3	0	-12
	1	0	-4	-6

   $P(3) = -6$

   b) $P(x) = 4x^3 + 2x^2 - 4x + 6 \quad c = -\frac{3}{2}$

$-\frac{3}{2}$	4	2	-4	6
	↓	-6	6	-6
	4	-4	2	0

   $P(-\frac{3}{2}) = 0$

2. Use the factor theorem to determine whether the binomial is a factor of the given polynomial.

   (P(c) = 0)

   a) $x - 1; \; P(x) = x^6 - x^4 + x^2 - 1$

   c = 1

1	1	0	-1	0	1	0	-1
	↓	1	1	0	0	1	1
	1	1	0	0	1	1	0

   yes, x-1 is a factor

   P(1) = 0

   b) $y + 2; \; P(y) = y^4 - y^2 + 4y + 2$

   c = -2

-2	1	0	-1	4	2
	↓	-2	4	-6	4
	1	-2	3	-2	6

   Not a factor

should give you remainder = 0

3. A root (solution) of the equation is given. Solve the equation.

* # of solutions should match highest degree polynomial

a) $x^3 + 3x^2 - 18x - 40 = 0$; 4

$$\begin{array}{r|rrrr} 4 & 1 & 3 & -18 & -40 \\ & \downarrow & 4 & 28 & 40 \\ \hline & 1 & 7 & 10 & 0 \end{array}$$

$x^2 + 7x + 10 = 0$ * can factor, complete the square, quadratic to solve

$(x+5)(x+2) = 0$

$x+5=0$, $x = -5$; $x+2=0$, $x = -2$

$x = 4, -5, -2$ (3 solutions)

don't forget the one they gave

b) $x^3 - 5x - 2 = 0$; -2

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & \downarrow & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & 0 \end{array}$$

$x^2 - 2x - 1 = 0$

$(x-1)(x+1) = 0$ — doesn't work, be careful when factoring: $x^2 - 1$ NOT $x^2 - 2x - 1$

$1x^2 - 2x - 1 = 0$ (a, b, c)

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$\frac{2 \pm \sqrt{8}}{2}$; $\sqrt{8}$: 4, 2 → 2, 2

$x = \frac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

$x = -2, 1 \pm \sqrt{2}$ (3 solutions)

Extended Practice

1. Use synthetic substitution for find P(c).

a) $P(x) = x^3 - 2x^2 - 5x - 7$ $c = 4$

$$\begin{array}{r|rrrr} 4 & 1 & -2 & -5 & -7 \\ & \downarrow & 4 & 8 & 12 \\ \hline & 1 & 2 & 3 & 5 \end{array}$$

$P(4) = 5$

b) $P(x) = 2x^3 + 3x^2 - 5x + 2$ $c = -3$

$$\begin{array}{r|rrrr} -3 & 2 & 3 & -5 & 2 \\ & \downarrow & -6 & 9 & -12 \\ \hline & 2 & -3 & 4 & -10 \end{array}$$

$P(-3) = -10$

c) $P(x) = 4x^3 - 4x^2 + 5x + 1$ $c = \frac{3}{2}$

$$\begin{array}{r|rrrr} \frac{3}{2} & 4 & -4 & 5 & 1 \\ & \downarrow & 6 & 3 & 12 \\ \hline & 4 & 2 & 8 & 13 \end{array}$$

$P(\frac{3}{2}) = 13$

d) $P(x) = 2x^4 - x^3 + x - 2$ $c = -\frac{3}{2}$

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 2 & -1 & 0 & 1 & -2 \\ & \downarrow & -3 & 6 & -9 & 12 \\ \hline & 2 & -4 & 6 & -8 & 10 \end{array}$$

$P(-\frac{3}{2}) = 10$

2. Use the factor theorem to determine whether the binomial is a factor of the given polynomial.

a) $x+1$; $P(x) = x^7 - x^5 + x^3 - x$

$c = -1$

$$\begin{array}{r|rrrrrrrr} -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ & \downarrow & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ \hline & 1 & -1 & 0 & 0 & 1 & -1 & 0 & \underline{0} \end{array}$$

yes x+1 is a factor

b) $y+1$; $P(y) = y^5 + y^4 + y^3 + y^2 + y + 1$

$c = -1$

$$\begin{array}{r|rrrrrr} -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & \downarrow & -1 & 0 & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & 0 & 1 & \underline{0} \end{array}$$

yes, y+1 is a factor

c) $z+2$; $P(z) = z^5 + 2z^4 + z^3 + 2z^2 + z + 2$

$c = -2$

$$\begin{array}{r|rrrrrr} -2 & 1 & 2 & 1 & 2 & 1 & 2 \\ & \downarrow & -2 & 0 & -2 & 0 & -2 \\ \hline & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

yes, z+2 is a factor

3. A root (solution) of the equation is given. Solve the equation.

a) $x^3 + 3x^2 - 3x - 9 = 0$; -3

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -3 & -9 \\ & \downarrow & -3 & 0 & 9 \\ \hline & 1 & 0 & -3 & 0 \end{array}$$

$x^2 - 3 = 0$

$+3 \quad +3$

$\sqrt{x^2} = \sqrt{3}$ $\quad x = \pm\sqrt{3}$

$X = -3, \sqrt{3}, -\sqrt{3}$

b) $2x^3 + 9x^2 + 7x - 6 = 0$; -2

$$\begin{array}{r|rrrr} -2 & 2 & 9 & 7 & -6 \\ & \downarrow & -4 & -10 & 6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$2x^2 + 5x - 3 = 0$
a b c

$x = \dfrac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)}$

$= \dfrac{-5 \pm \sqrt{49}}{4}$

$x = \dfrac{-5 \pm 7}{4} \rightarrow \dfrac{2}{4} = \dfrac{1}{2}$, $\dfrac{-12}{4} = -3$

$X = -2, \frac{1}{2}, -3$