Finding all Rational and Complex Zeros of Polynomial Functions

In this section we will take all that we have previously learned to find all of the rational and complex zeros of a polynomial function.

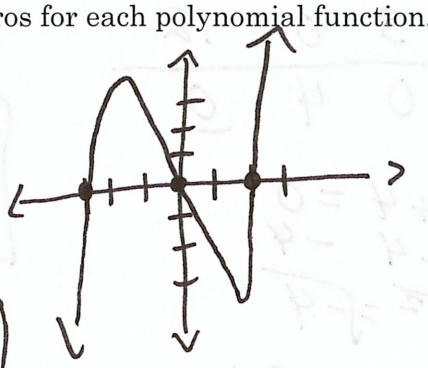
* graph first

Break for Practice: Find all of the rational and complex zeros for each polynomial function.

1. $y = x^3 + x^2 - 6x$

$x = -3, 0, 2$

# of zeros: 3



$-3 \mid 1 \;\; 1 \;\; -6 \;\; 0$

$\downarrow \;\; -3 \;\; 6$

$x^2 \;\; 0 \mid 1 \;\; -2 \;\; 0$

$\downarrow \;\; 0 \;\; 0$

$x \;\; 2 \mid 1 \;\; -2 \;\; 0$

$\downarrow \;\; 2$

$1 \;\; 0$ ✓

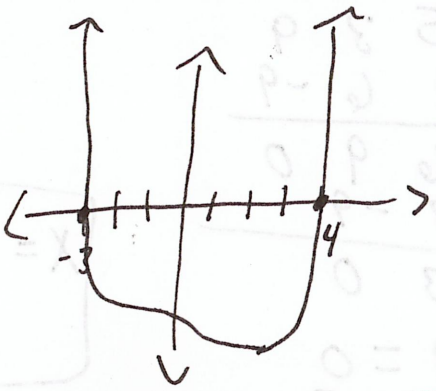
zeros: $-3, 0, 2$

all real

2. $y = x^4 - x^3 - 3x^2 - 9x - 108$

$x = -3, 4$

# of zeros: 4



$-3 \mid 1 \;\; -1 \;\; -3 \;\; -9 \;\; -108$

$\downarrow \;\; -3 \;\; 12 \;\; -27 \;\; 108$

$x^3 \;\; 4 \mid 1 \;\; -4 \;\; 9 \;\; -36 \;\; 0$

$\downarrow \;\; 4 \;\; 0 \;\; 36$

$x^2 \;\; 1 \;\; 0 \;\; 9 \;\; 0$

$x^2 + 9 = 0$

$-9 \;\; -9$

$\sqrt{x^2} = \sqrt{-9}$

$x = \pm 3i$

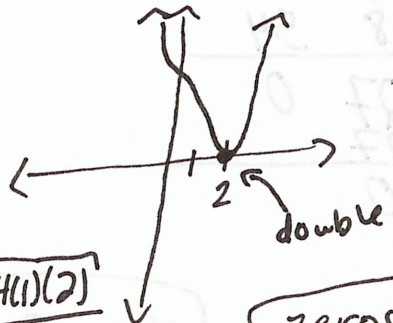
zeros: $-3, 4$ (real)

$3i, -3i$ (imaginary)

3. $y = x^4 - 6x^3 + 14x^2 - 16x + 8$

$x = 2, 2$

# of zeros: 4



$2 \mid 1 \;\; -6 \;\; 14 \;\; -16 \;\; 8$

$\downarrow \;\; 2 \;\; -8 \;\; 12 \;\; -8$

$x^3 \;\; 2 \mid 1 \;\; -4 \;\; 6 \;\; -4 \;\; 0$

$\downarrow \;\; 2 \;\; -4 \;\; 4$

$x^2 \;\; 1 \;\; -2 \;\; 2 \;\; 0$

$a \;\; b \;\; c$

$x^2 - 2x + 2 = 0$

$x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

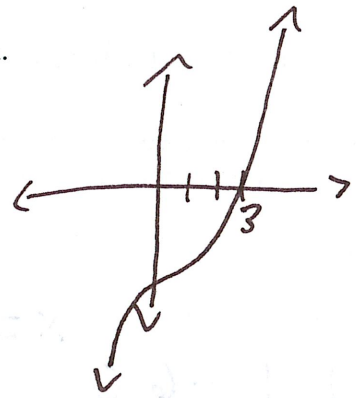
$= \dfrac{2 \pm 2i}{2} = 1 \pm i$

zeros: 2, 2 (real)

$1 \pm i$ (complex)

Extended Practice: Find all of the zeros for each polynomial function.

1. $y = x^3 - 3x^2 + 4x - 12$ $x = 3$



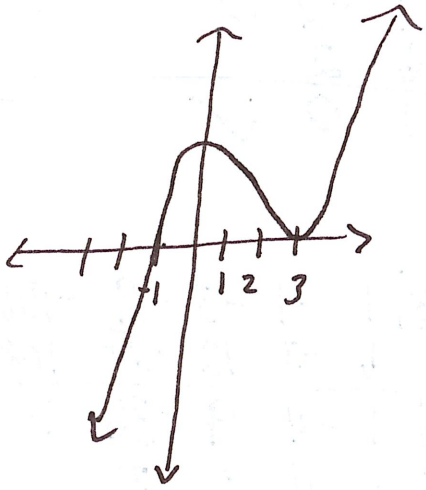
$$\begin{array}{r|rrrr} 3 & 1 & -3 & 4 & -12 \\ & \downarrow & 3 & 0 & 12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

x^2

$x^2 + 4 = 0$
$-4 \quad -4$
$\sqrt{x^2} = \sqrt{-4}$
$x = \pm 2i$

$x = 3, 2i, -2i$
real imag.

2. $y = x^3 - 5x^2 + 3x + 9$ $x = -1, 3, 3$

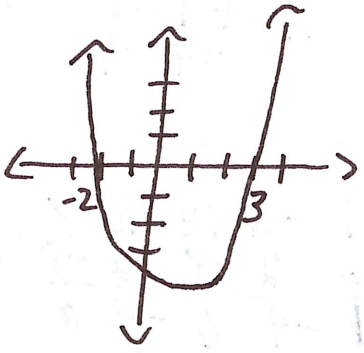


$$\begin{array}{r|rrrr} -1 & 1 & -5 & 3 & 9 \\ & \downarrow & -1 & 6 & -9 \\ \hline 3 & 1 & -6 & 9 & 0 \\ & \downarrow & 3 & -9 & \\ \hline & 1 & -3 & 0 & \end{array}$$

$x - 3 = 0$
$+3 \quad +3$
$x = 3$ ✓

$x = -1, 3, 3$
real

3. $y = x^4 - x^3 + 3x^2 - 9x - 54$ $x = -2, 3$



$$\begin{array}{r|rrrrr} -2 & 1 & -1 & 3 & -9 & -54 \\ & \downarrow & -2 & 6 & -18 & 54 \\ \hline x^3\ 3 & 1 & -3 & 9 & -27 & 0 \\ & \downarrow & 3 & 0 & 27 & \\ \hline x^2 & 1 & 0 & 9 & 0 & \end{array}$$

$x^2 + 9 = 0$
$-9 \quad -9$
$\sqrt{x^2} = \sqrt{-9}$
$x = \pm 3i$

$x = -2, 3$ (real)
$3i, -3i$ imaginary

4. $y = x^5 - 2x^4 + 40x^2 - 41x - 78$

$x = -3, -1, 2$

-3 | 1 −2 0 40 −41 −78
↓ −3 15 −45 15 78

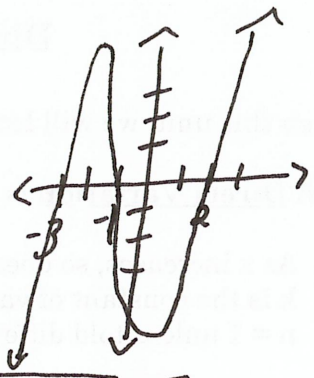
x^4 -1 | 1 −5 15 −5 −26 0
↓ −1 6 −21 26

x^3 2 | 1 −6 21 −26 0
↓ 2 −8 26

x^2 1 −4 13 0

a b c

$x^2 - 4x + 13 = 0$



$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$x = -3, -1, 2$ (real)

$2 \pm 3i$ complex

5. $y = -2x^6 + 5x^5 + 34x^4 - 16x^3 - 209x^2 - 376x + 240$

Try a viewing window of x-min = -5, x-max = 5, y-min = -1500, and y-max = 1000

x^6 x^5 x^4 x^3 x^2 x 0

$x = -3, 4, 4, \frac{1}{2}$?

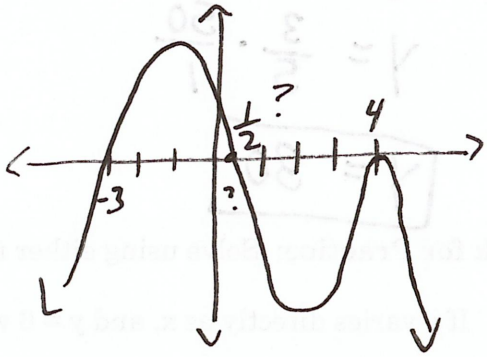
x^6 -3 | −2 5 34 −16 −209 −376 240
↓ 6 −33 −3 57 456 −240

x^5 4 | −2 11 1 −19 −152 80 0
↓ −8 12 52 132 −80

x^4 4 | −2 3 13 33 −20 0
↓ −8 −20 −28 20

$\frac{1}{2}$ | −2 −5 −7 5 0
↓ −1 −3 −5

x^2 −2 −6 −10 0 yes!



guess and check with $\frac{1}{2}$

$\frac{b}{a}$

$a \in \pm\{\pm 1, 2\}$

$$-2x^2 - 6x - 10 = 0$$

a b c

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(-2)(-10)}}{2(-2)}$$

$$= \frac{6 \pm \sqrt{-44}}{-4} = \frac{6 \pm 2i\sqrt{11}}{-4}$$

$$x = \frac{-3 \pm i\sqrt{11}}{2}$$

$\sqrt{-44}$

$i\sqrt{44}$

$4 \cdot 11$

$2i\sqrt{11}$

$x = -3, 4, 4, \frac{1}{2}, \frac{-3}{2} \pm \frac{i\sqrt{11}}{-2}$

Direct Variation and Proportion

To finish this unit, we will look at different types of variations.

Definition: A **Direct Variation** is a function in the form $y = kx^n, \; n > 0$

Notes: As x increases, so does y.
k is the constant of variation or proportion
n = 1 unless told differently

Example: If y varies directly as x, and y = 12 when x = 20, find y when x = 50.

Method 1

$y = k \cdot x$

$\frac{12}{20} = \frac{k \cdot 20}{20}$

$\frac{3}{5} = k$

$y = \frac{3}{5} \cdot \frac{50}{1}$

$\boxed{y = 30}$

Method 2 use proportions

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$

$\frac{12}{20} = \frac{y}{50}$ $\quad \frac{20y}{20} = \frac{600}{20}$

$y = 30$

Break for Practice: Solve using either method.

1. If y varies directly as x, and y = 6 when x = 4, find y when x = 12.

$\frac{6}{4} = \frac{y}{12}$

$\frac{72}{4} = \frac{4y}{4}$ $\quad \boxed{y = 18}$

2. If w varies directly as z, and w = 4.5 when z = 3, find z when w = 1.5.

$\frac{w}{3} = \frac{w}{z}$ $\quad \frac{4.5}{4.5} = \frac{4.5z}{4.5}$

$1 = z$

3. If p is directly proportional to q^3 , and p = 3 when q = 2, find p when q = 4.

$\frac{3}{8} = \frac{k(2^3)}{8}$

$\frac{3}{8} = k$

$p = \frac{3}{8}(4^3)$

$p = \frac{3}{8}\left(\frac{64}{1}\right)$

$p = 24$

or $\frac{3}{2^3} = \frac{p}{4^3}$

$\frac{192}{8} = \frac{8p}{8}$

$p = 24$

4. If x varies directly as 3y+2, and x = 10 when y = 6, find y when x = 7.

$x = k(3y+2)$

$10 = k(3(6)+2)$

$\frac{10}{20} = \frac{k(20)}{20}$

$\frac{1}{2} = k$

$7 = \frac{1}{2}(3y+2)$

$7 = \frac{3}{2}y + 1$

$-1 \quad -1$

$\frac{2}{3}(6) = \left(\frac{3}{2}y\right)\frac{2}{3}$

$4 = y$

5. If the sales tax on a \$38 purchase is \$2.85, what will the tax be on an \$84 purchase?

$\frac{\$38}{2.85} = \frac{\$84}{tax}$

$\frac{38x}{38} = \frac{239.4}{38}$

$x = \$6.30$

6. A survey showed that 52 out of 234 people questioned preferred hot cereal to cold. In a school of 1800 people, how many people are likely to prefer hot cereal?

$\frac{52}{234} = \frac{x}{1800}$

$\frac{93600}{234} = \frac{234x}{234}$

400 people = x

Extended Practice: Solve using either method

1. If y varies directly as x, and y = 6 when x = 15, find y when x = 25.

$y = kx \quad \frac{6}{15} = \frac{k \cdot 15}{15} \quad y = \frac{2}{5}\left(\frac{25}{1}\right)$

$\frac{2}{5} = k \quad \boxed{y = 10}$

2. If s is directly proportional to t, and s = 40 when t = 15, find t when s = 64.

$\frac{40}{15} = \frac{64}{t} \quad \frac{40t}{40} = \frac{960}{40}$

$\boxed{t = 24}$

3. If p is directly proportional to q, and p = 9 when q = 7.5, find q when p = 24.

$\frac{p}{q} \quad \frac{9}{7.5} = \frac{24}{q} \quad \frac{9q}{9} = \frac{180}{9}$

$\boxed{q = 20}$

4. If s varies directly as r^2, and s = 12 when r = 2, find s when r = 5.

$s = kr^2 \quad \frac{12}{4} = \frac{k(2^2)}{4} \quad s = 3(5^2)$

$3 = k \quad s = 3(25)$

$\boxed{s = 75}$

5. If y is directly proportional to $\sqrt{x}$, and y = 25 when x = 3, find x when y = 100.

$y = k\sqrt{x} \quad \frac{25}{\sqrt{3}} = \frac{k\sqrt{3}}{\sqrt{3}} \quad \frac{\sqrt{3}}{25}\left(\frac{100}{1}\right) = \left(\frac{25}{\sqrt{3}}\sqrt{x}\right)\frac{\sqrt{3}}{25}$

* keep radical

$\frac{25}{\sqrt{3}} = k \quad (4\sqrt{3})^2 = (\sqrt{x})^2$

$16 \cdot 3 = x \quad \boxed{x = 48}$

6. If w varies directly as 2x – 1, and w = 9 when x = 2, find x when w = 15.

$w = k(2x-1)$ $\quad 9 = k(2(2)-1)$ $\quad \frac{9}{3} = \frac{3k}{3}$ $\quad 3 = k$

$\frac{15}{3} = \frac{3(2x-1)}{3}$ $\quad 5 = 2x-1$ $\quad +1 \quad +1$ $\quad \frac{6}{2} = \frac{2x}{2}$ $\quad x = 3$

7. If the sales tax on a $60 purchase is $3.90, what would it be on a $280 purchase?

$\frac{60}{3.90} = \frac{280}{x}$ $\quad \frac{60x}{60} = \frac{1092}{60}$ $\quad x = \$18.20$

8. A real estate agent made a commission of $5400 on a house that sold at $120,000. At this rate, what commission will the agent make on a house that sells for $145,000?

$\frac{5400}{120,000} = \frac{x}{145,000}$ $\quad \frac{120,000x}{120,000} = \frac{783,000,000}{120,000}$ $\quad x = \$6525$

9. The acceleration of an object varies directly as the force acting on it. If a force of 240 newtons causes an acceleration of 150 m/s², what force will cause an acceleration of 100 m/s².

a = acceleration
f = force

$a = k \cdot f$ $\quad \frac{150}{240} = \frac{k(240)}{240}$ $\quad \frac{5}{8} = k$

$\frac{8}{8}\left(\frac{100}{1}\right) = \left(\frac{5}{8}f\right)\frac{8}{5}$ $\quad 160 \text{ newtons} = f$

10. A public-opinion poll found that of a sample of 450 voters, 252 favored a school bond measure. If 20,000 people vote, about how many are likely to vote for the bond measure?

$\frac{\text{favored}}{\text{total}}$ $\quad \frac{252}{450} = \frac{x}{20,000}$ $\quad \frac{450x}{450} = \frac{5040000}{450}$ $\quad x = 11,200 \text{ votes}$

Inverse and Joint Variations

In the last section we learned what a direct variation is. Now we will learn what an inverse variation is.

Definition: An **Inverse Variation** is a function in the form $y = \frac{k}{x}$,
Note: As x increases, y will decrease.

Example: If y varies inversely as x, and y = 5 when x = 4, find x when y = 10.

$y = \frac{k}{x}$ $\quad 4(5) = \left(\frac{k}{4}\right)4$ $\quad \frac{10}{1} = \frac{20}{x}$

$20 = k$ $\quad \frac{10x}{10} = \frac{20}{10}$

$\boxed{x = 2}$

Break for Practice: Solve

1. If a is inversely proportional to b, and b = 12 when a = 8, find b when a = 3.

$a = \frac{k}{b}$ $\quad 12(8) = \frac{k}{12} \cdot 12$ $\quad \frac{3}{1} = \frac{96}{b}$ $\quad \frac{3b}{3} = \frac{96}{3}$

$96 = k$

$\boxed{b = 32}$

2. If x varies inversely as the square of y, and x = 2 when y = 12, find y when x = 8

$x = \frac{k}{y^2}$ $\quad 144(2) = \left(\frac{k}{12^2}\right)144$ $\quad \frac{8}{1} = \frac{288}{y^2}$ $\quad \sqrt{y^2} = \sqrt{36}$

$288 = k$ $\quad \frac{8y^2}{8} = \frac{288}{8}$ $\quad \boxed{y = \pm 6}$

It is possible to work with more than one variation at a time. If the term **Jointly** is used, then it means that several variables are varying **directly**.

Break for Practice: Solve

1. If x varies jointly as y and z, and x = 100 when y = 20 and z = 10, find x when y = 60 and z = 30.

$x = k(y \cdot z)$ $\quad \frac{100}{200} = \frac{k(20 \cdot 10)}{200}$ $\quad x = \frac{1}{2}(60)(30)$

$\frac{1}{2} = k$ $\quad \boxed{x = 900}$