c) $log_5 625 = 4$ $\quad 5^4 = 625$

d) $log\, 200 = x$ $\quad 10^x = 200$

3. Solve each of the following.

a) $log_5 x = 2$

$5^2 = x$

$\boxed{25 = x}$

b) $log_7 x = -2$

$7^{-2} = x$

$\frac{1}{7^2} = x$

$\boxed{\frac{1}{49} = x}$

c) $log_x 36 = 2$

$\sqrt{x^2} = \sqrt{36}$

$x = 6$

d) $log_x 8 = \frac{3}{2}$

$\left(x^{3/2}\right)^{2/3} = (8)^{2/3}$

$x = \left(\sqrt[3]{8}\right)^2$

$x = 2^2 = 4$

e) $log_{12} 144 = x$

$12^x = 144$

$12^x = 12^2$

$x = 2$

f) $log_3 \left(\frac{1}{27}\right) = x$

$3^x = \frac{1}{27}$

$3^x = 3^{-3}$

$\boxed{x = -3}$

Extended Practice:

1. Complete each chart

Log form	$log_2 32 = 5$	$log_3 9=2$	$log_7 \sqrt{7} = \frac{1}{2}$	$log_3 \frac{1}{81} = -4$
Exponential Form	$2^5 = 32$	$3^2 = 9$	$7^{1/2} = \sqrt{7}$	$3^{-4} = \frac{1}{81}$

Log form	$\log_4 64 = 3$	$\log_9 27 = \frac{3}{2}$	$\log_{10} 0.01 = -2$	$\log_{16} \frac{1}{8} = \frac{-3}{4}$
Exponential Form	$4^3 = 64$	$9^{\frac{3}{2}} = 27$	$10^{-2} = 0.01$	$16^{\frac{-3}{4}} = \frac{1}{8}$

2. Solve for x in each equation

a) $log_6 6 = x$

$6^x = 6^1$

$x = 1$

b) $log_3 1 = x$

$3^x = 1$

$\boxed{x = 0}$

c) $log_5 125 = x$

$5^x = 125$

$5^x = 5^3$

$\boxed{x = 3}$

d) $log_2\left(\frac{1}{8}\right) = x$

$2^x = \frac{1}{8}$

$2^x = 2^{-3}$

$\boxed{x = -3}$

e) $log_7 x = 2$

$7^2 = x$

$49 = x$

f) $log_6 x = 3$

$6^3 = x$

$\boxed{216 = x}$

g) $log_9 x = -\frac{1}{2}$

$9^{-\frac{1}{2}} = x$

$\frac{1}{9^{1/2}} = x$

$\frac{1}{\sqrt{9}} = x$

$\boxed{\frac{1}{3} = x}$

h) $log_6 x = 2.5$

$6^{2.5} = x$

$\boxed{88.18 = x}$

i) $log_x 27 = \frac{3}{2}$

$(x^{3/2})^{2/3} = (27)^{2/3}$

$x = (\sqrt[3]{27})^2$

$x = 3^2 = 9$

j) $log_x 64 = 6$

$\sqrt[6]{x^6} = \sqrt[6]{64}$

$\boxed{x = 2}$

k) $log_x 7 = -\frac{1}{2}$

$x^{-\frac{1}{2}} = 7$

$\frac{1}{\sqrt{x}} = 7$

$\frac{1}{7} = \frac{7\sqrt{x}}{7}$

$\left(\frac{1}{7}\right)^2 = (\sqrt{x})^2$

$\boxed{x = \frac{1}{49}}$

l) $log_x 7 = 1$

$x^1 = 7$

$\boxed{x = 7}$

Properties of Logs

Just as we had special properties, or laws, of exponents, we also have special properties of logarithms. The properties of logs are related to the laws of exponents.

Example: Property: $log_b(xy) = log_b x + log_b y$

Verify this property on your calculator with the example: $\log(9 \cdot 5) = \log 9 + \log 5$

Now follow along with the proof of this property:

Let $r = \log_b x$ and

$s = \log_b y$

so $b^r = x$, $b^s = y$

$xy = b^{r+s}$

$\log_b(xy) = r + s$ and

$\log_b(xy) = \log_b x + \log_b y$

The other properties can be proved in a similar fashion if you wish.

Log Property	Related Exponent Property	Example to verify property
$log_b(xy) = \log_b x + \log_b y$	$x^a \cdot x^b = x^{a+b}$	Already done
$log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\frac{x^a}{x^b} = x^{a-b}$	$\log(51 \div 3) = \log 51 - \log 3$ $1.2304 = 1.7076 - .4772$ ✓
$log_b x^n = n \log_b x$	$(x^a)^b = x^{ab}$	$\log 5^4 = 4 \cdot \log 5$ $2.7959 = 4(.6989)$ ✓

Break for Practice: Using the properties and these given facts, find the following values without using the log key on the calculator.

<u>Given Facts</u>: log 5 = 0.6990 and log 3 = 0.4771 Verify that log 10 = 1

Evaluate:

1. log 15 = $\log(5 \cdot 3) = \log 5 + \log 3$
   $= .6990 + .4771 = \boxed{1.1761}$
2. log 30 = $\log(3 \cdot 10) = \log 3 + \log 10 = .4771 + 1 = \boxed{1.4771}$
3. log 300 = $\log(3 \cdot 100) = \log(3 \cdot 10^2) = \log 3 + \log 10^2$
   $= \log 3 + 2 \log 10 = .4771 + 2(1)$
   $= \boxed{2.4771}$
4. log 2 = $\log\left(\frac{10}{5}\right) = \log 10 - \log 5$
   $= 1 - .6990 = \boxed{.301}$
5. log 25 = $\log 5^2 = 2 \log 5$
   $= 2(.6990) = \boxed{1.398}$
6. log √3 = $\log 3^{1/2} = \frac{1}{2} \log 3$
   $= \frac{1}{2}(.4771) = \boxed{.23855}$

Extended Practice: Using the properties and these given facts, find the following values without using the log key on the calculator.

Given Facts: log 9 = 0.95 and log 2 = 0.30 remember that log 10 = 1

1. $\log 81 = \log(9^2) = 2\log 9 = 2(.95) = \boxed{1.9}$

2. $\log \frac{9}{2} = \log 9 - \log 2 = .95 - .30 = \boxed{.65}$

3. $\log \sqrt{2} = \log 2^{1/2} = \frac{1}{2}\log 2 = \frac{1}{2}(.30) = \boxed{.15}$

4. $\log 3 = \log\sqrt{9} = \log 9^{1/2} = \frac{1}{2}\log 9 = \frac{1}{2}(0.95) = \boxed{.475}$

5. $\log 8 = \log 2^3 = 3\log 2 = 3(.30) = \boxed{.90}$

6. $\log 36 = \log(9\cdot 2^2) = \log 9 + \log 2^2 = \log 9 + 2\log 2$
$.95 + 2(.30) = \boxed{1.55}$

7. $\log \frac{20}{9} = \log\left(\frac{10\cdot 2}{9}\right) = \log 10 + \log 2 - \log 9$
$1 + .30 - .95 = \boxed{.35}$

8. $\log 900 = \log(9\cdot 10^2) = \log 9 + \log 10^2 = \log 9 + 2\log 10$
$= .95 + 2(1) = \boxed{2.95}$

9. $\log \frac{1}{9} = \log(9^{-1}) = -1\cdot\log 9$
$= -1(.95) = \boxed{-.95}$

10. $\log \frac{1}{2000} = \log(2\cdot 10^3)^{-1} = -1(\log 2 + \log 10^3) = -1(\log 2 + 3\log 10)$
$-1(.30 + 3(1)) = \boxed{-3.30}$

11. $\log \sqrt[3]{\frac{2}{9}} = \log 2^{1/3}\cdot 9^{-1/3} = \log 2^{1/3} + -1(\log 9^{1/3})$
$= \frac{1}{3}\log 2 - \frac{1}{3}\log 9 = \frac{1}{3}(.30) - \frac{1}{3}(.95)$
$.1 - .3167 = \boxed{-.2167}$

12. $\log 162 = \log(9^2\cdot 2)$
$= \log 9^2 + \log 2$
$= 2\log 9 + \log 2$
$= 2(.95) + .30$
$= \boxed{2.2}$

In the previous problems, we expanded single logs. Now we want to do the reverse process and combine logs together.

Break for Practice: Use the properties of logs to bring the following together as a single log.

1. $log_8 5 + log_8 7 =$ $\log_8(5\cdot 7) = \log_8 35$

2. $log_3 75 - log_3 15 =$ $\log_3\left(\frac{75}{15}\right) = \log_3 5$

3. $4 \cdot log_7 2 =$ $\log_7 2^4 = \log_7 16$

4. $log_6 28 - log_6 4 + log_6 7 =$ $\log_6\left(\frac{28}{4}\right) + \log_6 7$
$= \log_6 7 + \log_6 7 = \log_6(7\cdot 7) = \boxed{\log_6 49}$

5. $5 \cdot log_6 2 - log_6 16 =$ $\log_6 2^5 - \log_6 16 = \log_6\left(\frac{32}{16}\right) = \boxed{\log_6 2}$

6. $3 \cdot log_5 2 + \frac{1}{3}[log_5 8 + log_5 8] - log_5 4 =$

$3\cdot\log_5 2 + \frac{1}{3}[\log_5 64] - \log_5 4$

$\log_5 2^3 + \log_5 64^{1/3} - \log_5 4$

$\log_5 8 + \log_5 4 - \log_5 4$

$\log_5(8\cdot 4) - \log_5 4$

$\log_5(32) - \log_5 4$

$\log_5\left(\frac{32}{4}\right) = \boxed{\log_8 4}$

We can also use logs to work with numbers too large for the calculator.

Find 385^{92}

$385^{92} = x$

*log both sides

$\log 385^{92} = \log x$

$92 \log 385 = \log x$

*convert back to exponential form

$237.8624 = \log x$

$10^{237.8624} = x$

$10^{237} \cdot 10^{.8624} = x$

$\boxed{7.2845 \times 10^{237} = x}$

Extended Practice: Use the properties of logs to bring the following together as a single log.

1. $log_a 3 + log_a 4 =$ $\log_a(3\cdot4) = \boxed{\log_a 12}$

2. $log_a 7 - log_a 5 =$ $\boxed{\log_a\left(\frac{7}{5}\right)}$

3. $4 \cdot log_a 2 =$ $\log_a 2^4 = \boxed{\log_a 16}$

4. $2 \cdot log_a 9 =$ $\log_a 9^2 = \boxed{\log_a 81}$

5. $\frac{1}{2} \cdot log_a 36 =$ $\log_a 36^{1/2} = \log_a \sqrt{36} = \boxed{\log_a 6}$

6. $-log_a \frac{1}{6} =$ $\log_a\left(\frac{1}{6}\right)^{-1} = \boxed{\log_a 6}$

7. $log_b 3 + log_b 5 + log_b 2 =$ $\log_b(3\cdot5) + \log_b 2$
$\log_b(15\cdot2) = \boxed{\log_b 30}$

8. $log_b 6 + log_b 5 - log_b 2 =$ $\log_b(6\cdot5) - \log_b 2$
$= \log_b\left(\frac{30}{2}\right) = \boxed{\log_b 15}$

9. $2 \cdot log_b p + log_b q =$ $\log_b p^2 + \log_b q = \boxed{\log_b(p^2\cdot q)}$

10. $log_b x - 3 \cdot log_b y =$ $\log_b x + \log_b y^{-3} = \boxed{\log_b\left(\frac{x}{y^3}\right)}$

11. $\frac{1}{2} \cdot log_b r + \frac{1}{2} \cdot log_b s =$ $\log_b r^{1/2} + \log_b s^{1/2} = \log_b(rs)^{1/2} = \boxed{\log_b \sqrt{rs}}$

12. $\frac{1}{2}(log_b x - log_b y) =$ $(\log_b x - \log_b y)^{1/2}$
$= \log_b\left(\frac{x}{y}\right)^{1/2} = \boxed{\log_b \sqrt{\frac{x}{y}}}$

13. Find the value of 431^{75}.

$$431^{75} = x$$
$$\log 431^{75} = \log x$$
$$75 \log 431 = \log x$$
$$197.5858 = \log x$$

$$10^{197.5858} = x$$
$$10^{.5858} \cdot 10^{197} = x$$
$$\boxed{3.8530 \times 10^{197} = x}$$

Change of Base Formula

In this section we will see how to use the calculator for logarithms of any base.

Challenge: Find $\log_2 7 = x$

$$2^x = 7$$
$$\log 2^x = \log 7$$
$$\frac{x \cdot \log 2}{\log 2} = \frac{\log 7}{\log 2}$$

$x = 2.807$

$\log_2 7 = 2.807$

This example illustrates the change of base formula.

Change of Base Formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$

* usually use $a = 10$

Break for Practice: Use the change of base formula to evaluate the following.

1. $\log_5 28 = \dfrac{\log 28}{\log 5} = 2.070$

2. $\log_{11} 412 = \dfrac{\log 412}{\log 11} = 2.511$

3. $\log_{48} 0.015 = \dfrac{\log .015}{\log 48} = -1.085$

4. Simplify each of the following.

   a) $\log_b b = \dfrac{\log b}{\log b} = 1$ (Remember this property. It will be very useful.)

   *

b) $log_{82}82^7 =$ $7 \cdot \log_{82} 82 = 7\left(\frac{\log 82}{\log 82}\right) = 7 \cdot 1 = \boxed{7}$

*rewrite in log form

c) $12^{\log_{12}5} = X$ $\underline{\log_{12}} X = \underline{\log_{12}} 5$ same base

$\boxed{X = 5}$

Extended Practice:

1. Use the change of base formula to evaluate the following.

a) $log_7 28 =$ $\frac{\log 28}{\log 7} = 1.7124$

b) $log_4 563 =$ $\frac{\log 563}{\log 4} = 4.5685$

c) $log_3 0.58 =$ $\frac{\log .58}{\log 3} = -.4958$

d) $log_8 0.0039 =$ $\frac{\log .0039}{\log 8} = -2.6674$

e) $log_5 78{,}125 =$ $\frac{\log 78125}{\log 5} = 7$

f) $log_7 5{,}764{,}801 =$ $\frac{\log 5{,}764{,}801}{\log 7} = 8$

g) $log_{0.3} 58 =$ $\frac{\log 58}{\log .3} = -3.3725$

h) $log_2 13.7 =$ $\frac{\log 13.7}{\log 2} = 3.7761$

i) $log_4(-22.4) =$ $\frac{\log(-22.4)}{\log 4} = \emptyset$ * you can't take the log of a negative

j) $log_{0.6} 0.36 =$ $\frac{\log .36}{\log .6} = 2$

2. Simplify each expression

a) $log_{49}49^8 =$ $8 \cdot \log_{49} 49 = 8\left(\frac{\cancel{\log 49}}{\cancel{\log 49}}\right) = 8$

b) $log_{64}64^{\frac{5}{6}} =$ $\frac{5}{6}(\log_{64} 64) = \boxed{\frac{5}{6}}$

c) $3 \cdot log_{24}24^{\frac{-2}{3}} =$ $\frac{\cancel{3}}{1} \cdot \left(\frac{-2}{\cancel{3}}\right) \cancel{\log_{24} 24} = \boxed{-2}$

d) $5 \cdot log_{37}37^9 =$ $5 \cdot 9 \cdot \cancel{\log_{37} 37} = \boxed{45}$

e) $10^{log9} = X$ $\quad \log_{10} X = \log_{10} 9$ $\quad \boxed{X = 9}$

f) $6^{log_6 13} = X \rightarrow \log_6 X = \log_6 13$ $\quad \boxed{X = 13}$

g) $3^{log_3 7} = X$ $\quad \log_3 X = \log_3 7 \Rightarrow \boxed{7 = X}$

h) $10^{log2001} = X$ $\quad \log_{10} X = \log_{10} 2001 \Rightarrow \boxed{X = 2001}$

Solving Equations Involving Exponents and Logarithms

A variety of equations can be solved using exponents and logarithms. On these problems we will be working with two different bases. Both of these can be done with the calculator.

log ⟶ common log, base 10

ln ⟶ natural log, base e $\qquad e \approx 2.718$

By definition e is the limit of the sequence of terms $\left(1 + \frac{1}{x}\right)^x$ as $x \to \infty$

* $\log_e = \ln$ $\qquad \ln_e$