Algebra II
Unit 9
Exponential and Logarithmic Functions

Unit "I can" statements:

1. I can simplify expressions involving fractional exponents, and convert between exponential and radical forms.
2. I can simplify expressions involving irrational exponents.
3. I can solve exponential equations.
4. I can evaluate composite functions and find the inverse of a function.
5. I can simplify expressions and solve equations by converting between exponential and logarithmic forms.
6. I can apply the basic properties of logarithms.
7. I can evaluate logarithms in any given base by applying the change of base formula.
8. I can find equations for exponential functions when given two points.
9. I can solve applications involving exponential functions.

Common Core State Standards that are addressed in this unit include: A.SSE.1b, A.SSE.2a, A.CED.1a, A.CED.2a, F.IF.7e
For more information see www.corestandards.org/Math/

Rational Exponents

In this unit, we will learn about exponential and logarithmic functions. This means that we will need to expand our knowledge of the laws of exponents. Let's quickly review what we already know.

Review:

Example	Exponent Law
$(2a^2b)(3a^3b^4) = 6a^5b^5$	$x^a \cdot x^b = x^{a+b}$
$\dfrac{25a^4b^6}{5ab^4} = 5a^3b^2$	$\dfrac{x^a}{x^b} = x^{a-b}$
$(a^2b^3)^4 \quad a^8b^{12}$	$(x^a)^b = x^{ab}$
$a^{-2} = \dfrac{1}{a^2}$	$x^{-a} = \dfrac{1}{x^a}$
$a^0 = 1$	$x^0 = 1$

Question: What do you suppose $b^{\frac{1}{2}}$ means?

$(b^{1/2})^2 = b^{1/2 \cdot 2} = b^1$

$(\sqrt{b})^2 = b$ so $b^{1/2} = \sqrt{b}$

Definition of $b^{\frac{p}{q}}$ exponential form $\longrightarrow \underbrace{\sqrt[q]{b^p} \text{ or } \left(\sqrt[q]{b}\right)^p}_{\text{radical form}}$

Break for Practice:

1. Simplify

a) $16^{\frac{1}{2}} = \sqrt{16} = 4$

b) $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3$

$3^3 = 27$

c) $36^{\frac{-3}{2}} = \frac{1}{36^{3/2}} = \frac{1}{(\sqrt{36})^3} = \frac{1}{6^3}$

$\boxed{\frac{1}{216}}$

d) $(-64)^{\frac{2}{3}} = \left(\sqrt[3]{(-64)}\right)^2$

$(-4)^2 = 16$

e) $4^{3.5} = 4^{3\frac{1}{2}} = 4^{7/2} = (\sqrt{4})^7$

$2^7 = \boxed{128}$

f) $(8^5)^{\frac{1}{3}} = 8^{\frac{5}{3}} = (\sqrt[3]{8})^5$

$2^5 = \boxed{32}$

2. Rewrite in exponential form.

a) $\sqrt{x^5y^4} = x^{\frac{5}{2}}y^{\frac{4}{2}} = \boxed{x^{\frac{5}{2}}y^2}$

b) $\sqrt[5]{x^4y^{-6}}$ $\quad x^{\frac{4}{5}}y^{\frac{-6}{5}}$

c) $\sqrt[4]{\frac{3^4a^{-6}}{b^2}} = \frac{3^{\frac{4}{4}}a^{\frac{-6}{4}}}{b^{\frac{2}{4}}} = \frac{3a^{-3/2}}{b^{1/2}} = \boxed{3a^{-3/2}b^{-1/2}}$

or

$\sqrt[4]{3^4a^{-6}b^{-2}}$

3. Solve each equation.

a) $\frac{6x^{\frac{2}{3}}}{6} = \frac{54}{6}$

$(x^{2/3})^{3/2} = (9)^{3/2}$

$x = (\sqrt{9})^3$

$x = 3^3$

$\boxed{x = 27}$

b) $(t-4)^{\frac{2}{5}} - 3 = 1$

$\quad +3 \quad +3$

$\left[(t-4)^{2/5}\right]^{5/2} = (4)^{5/2}$

$t-4 = (\sqrt{4})^5$

$t-4 = 2^5$

$t-4 = 32$

$\quad +4 \quad +4$

$\boxed{t = 36}$

Extended Practice:

1. Simplify

a) $81^{\frac{1}{2}} = \sqrt{81} = \boxed{9}$

b) $49^{\frac{-1}{2}} = \frac{1}{49^{\frac{1}{2}}} = \frac{1}{\sqrt{49}} = \boxed{\frac{1}{7}}$

c) $4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = \boxed{8}$

d) $16^{\frac{3}{4}} = (\sqrt[4]{16})^3$ $2^3 = \boxed{8}$

e) $(-125)^{\frac{-1}{3}} = \frac{1}{(-125)^{1/3}} = \frac{1}{\sqrt[3]{-125}} = \boxed{\frac{1}{-5}}$

f) $4^{-0.5} = 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}$

g) $-8^{\frac{2}{3}} = (\sqrt[3]{-8})^2$ $(-2)^2 = \boxed{4}$

h) $\left(5^{\frac{1}{3}}\right)^{-3} = 5^{-\frac{3}{3}} = 5^{-1} = \boxed{\frac{1}{5}}$

2. Rewrite in exponential form.

a) $\sqrt{x^3y^5} = x^{\frac{3}{2}}y^{\frac{5}{2}}$

b) $\sqrt[3]{p^4q} = p^{\frac{4}{3}}q^{\frac{1}{3}}$

c) $\sqrt[3]{8b^6c^{-4}}$

$2b^{\frac{6}{3}}c^{\frac{-4}{3}}$

$\boxed{2b^2c^{-4/3}}$

3. Solve each equation

a) $\left(a^{\frac{3}{4}}\right)^{4/3} = (8)^{4/3}$

$a = (\sqrt[3]{8})^4$

$a = 2^4$

$\boxed{a = 16}$

b) $\left((3x+1)^{\frac{3}{4}}\right)^{4/3} = (8)^{4/3}$

$3x+1 = (\sqrt[3]{8})^4$ 2^4

$3x+1 = 16$

$-1 \quad -1$

$\frac{3x}{3} = \frac{15}{3}$

$\boxed{x = 5}$

c) $\left[(8-y)^{\frac{1}{3}}\right]^3 = 4^3$

$8 - y = 64$

$-8 \quad -8$

$\frac{-y}{-1} = \frac{56}{-1}$

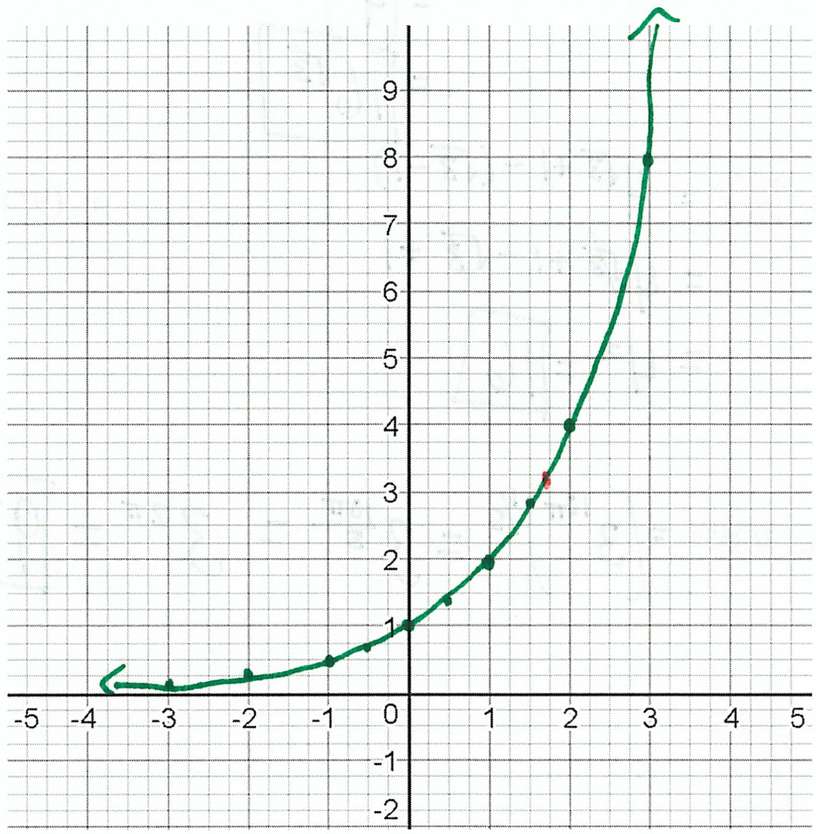
$\boxed{y = -56}$

Real Number Exponents

In this section, we will extend our meaning of exponents to include any real number.

Graph: $y = 2^x$

x	$y = 2^x$	point
0	$2^0 = 1$	$(0,1)$
1	$2^1 = 2$	$(1,2)$
2	$2^2 = 4$	$(2,4)$
3	$2^3 = 8$	$(3,8)$
-1	$2^{-1} = \frac{1}{2}$	$(-1, \frac{1}{2})$
-2	$2^{-2} = \frac{1}{4}$	$(-2, \frac{1}{4})$
-3	$2^{-3} = \frac{1}{8}$	$(-3, \frac{1}{8})$
$\frac{1}{2}$	$2^{1/2} = \sqrt{2}$	$(\frac{1}{2}, 1.4)$
$-\frac{1}{2}$	$2^{-1/2} = \frac{1}{\sqrt{2}}$	$(-\frac{1}{2}, .7)$
$\frac{3}{2}$	$2^{3/2} = (\sqrt{2})^3$	$(\frac{3}{2}, 2.8)$



This is an example of an exponential function. Use your graph to estimate the value of $2^{\sqrt{3}}$.

$\sqrt{3} \approx 1.7 = 2^{1.7}$ ≈ 2.25 actual $2^{\sqrt{3}} \approx 2.32\ldots$

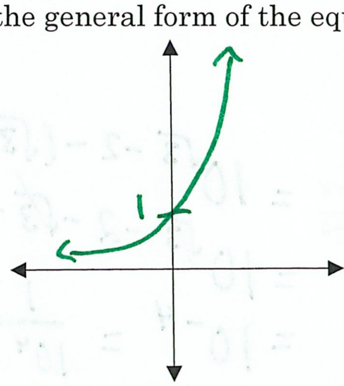
Definition: An **<u>Exponential Function</u>** is in the form $y = b^x$. $b > 0$ *but* $b \neq 1$.

Use the graphing calculator/computer to look at graphs of $y = 2^x$, $y = 3^x, y = 4^x$.

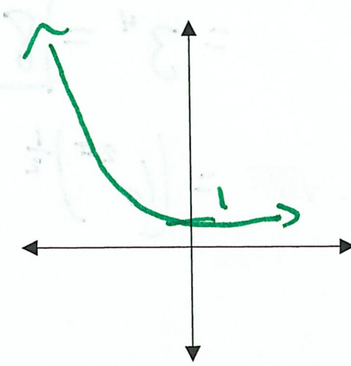
Also look at graphs of $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, y= $\left(\frac{1}{4}\right)^x$

Result: Sketch the general form of the equation $y = b^x$ if

a) b > 0



b) 0 < b < 1



Break for Practice: Simplify

1. $4^{\sqrt{3}} \cdot 4^{\sqrt{3}} = 4^{\sqrt{3}+\sqrt{3}} = 4^{2\sqrt{3}}$
$= (4^2)^{\sqrt{3}}$
$= \boxed{16^{\sqrt{3}}}$

2. $\left(4^{\sqrt{3}}\right)^{\sqrt{3}} = 4^3 = \boxed{64}$

3. $\frac{4^{\sqrt{3}+1}}{4^{\sqrt{3}-1}} = 4^{\sqrt{3}+1-(\sqrt{3}-1)}$
$= 4^{\cancel{\sqrt{3}}+1-\cancel{\sqrt{3}}+1}$
$= 4^2 = \boxed{16}$

4. $(9^{\pi})^2 = 9^{2\pi} = \boxed{81^{\pi}}$

5. $\sqrt[5]{3^{10\pi}} = \left(3^{10\pi}\right)^{1/5} = 3^{\frac{10\pi}{5}} = 3^{2\pi} = \boxed{9^{\pi}}$

Extended Practice: Simplify

1. $3^{\sqrt{2}} \cdot 3^{\sqrt{2}} = 3^{2\sqrt{2}}$
$= \boxed{9^{\sqrt{2}}}$

2. $\left(3^{\sqrt{2}}\right)^2 = 3^{2\sqrt{2}}$
$= \boxed{9^{\sqrt{2}}}$

3. $\left(3^{\sqrt{2}}\right)^{\sqrt{2}} = 3^2 = \boxed{9}$

4. $\frac{3^{\sqrt{2}+2}}{3^{\sqrt{2}-2}} = 3^{\sqrt{2}+2-(\sqrt{2}-2)}$
$= 3^{\cancel{\sqrt{2}}+2-\cancel{\sqrt{2}}+2}$
$= 3^4 = \boxed{81}$

5. $(10^{\pi})^2 = 10^{2\pi} = \boxed{100^{\pi}}$

6. $\sqrt{6^{2\pi}} = \left(6^{2\pi}\right)^{\frac{1}{2}} = 6^{\frac{2\pi}{2}} = \boxed{6^{\pi}}$

7. $\frac{10^{\sqrt{3}-2}}{10^{\sqrt{3}+2}} = 10^{\sqrt{3}-2-(\sqrt{3}+2)}$
$= 10^{\cancel{\sqrt{3}}-2-\cancel{\sqrt{3}}-2}$
$= 10^{-4} = \frac{1}{10^4} = \boxed{\frac{1}{10000}}$