**Algebra II**

**Unit 10**

**Sequences and Series**

Unit “I can” statements:

1. I can determine if a sequence is arithmetic, geometric, or neither, and I can extend a sequence.
2. I can find the formula for an arithmetic sequence, find a specified term within an arithmetic sequence, and find a given number of arithmetic means between two terms in an arithmetic sequence.
3. I can find the formula for a geometric sequence, find a specified term within a geometric sequence, and find a given number of geometric means between two terms in a geometric sequence.
4. I can write series in expanded form and in summation (sigma) notation.
5. I can find partial sums of arithmetic and geometric series.

Common Core State Standards that are addressed in this unit include: A.SSE.1a, A.SSE.4b, A.CED.2a

For more information see [www.corestandards.org/Math/](http://www.corestandards.org/Math/)

Introduction to Sequences

 In this, the last unit, we will study sequences and series. Sequences are simply a type of pattern.

**Definition**: A **Sequence** is a function whose domain is the set of natural (counting) numbers, and whose range is the set of term values.

 **Example**: Consider the sequence 3, 5, 7, 9, … can be written in a table as

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Term number (x)** | 1 | 2 | 3 | 4 | … |
| **Term value (y)** |  |  |  |  |  |

 The notation that is used is this.

 t1 = 3

 t2  =

 t3 =

 t4 =

 tn =

 There are many special types of sequences. We will mainly concentrate on two.

**Arithmetic Sequence** – all terms are separated by a common , d.

 Example:

**Geometric Sequence** – all terms are separated by a common , r.

 Example

**Break for Practice:**

1. Identify the following as arithmetic, geometric, or neither. Then fill in the missing terms.
2. 7, 12, 17, 22, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_\_
3. 2, -4, 8, -16, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
4. $\frac{1}{2} , \frac{1}{3} , \\_\\_\\_\\_\\_\\_\\_ \frac{1}{5} , \frac{1}{6} ,\\_\\_\\_\\_\\_\\_\\_\\_\\_$
5. -1, 2, -3, 4, \_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_
6. 21, 15, 9, 3, \_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_
7. 5, 15, 45, 135, \_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_
8. Find the first 4 terms, and identify the sequence as arithmetic, geometric, or neither.
9. $t\_{n}=1-2n$
10. $t\_{n}=2\left(3^{n}\right)$
11. $t\_{n}=\frac{1}{n^{2}}$
12. Find the next two terms by looking at the pattern in the difference between terms.
13. 8, 9, 11, 14, \_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_\_
14. 5, 7, 11, 17, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_

**Extended Practice:**

1. Identify the following as arithmetic, geometric, or neither. Then fill in the missing terms.
2. 20, 17,14, 11, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
3. 5, 9, 13, 17, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
4. 1, 5, 25, 125, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
5. 256, 64, 16, 4, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
6. 18, 22, 26, \_\_\_\_\_\_\_\_\_\_ , 34, \_\_\_\_\_\_\_\_\_\_
7. 4, \_\_\_\_\_\_\_\_\_\_ , -4, -8, -12, \_\_\_\_\_\_\_\_\_\_
8. $1, \frac{1}{4} , \frac{1}{9} , \frac{1}{16} ,$ \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
9. 32, -16, 8, -4, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
10. Find the first four terms of the sequence with the given formula. Then tell whether the sequence is arithmetic, geometric, or neither.
11. $t\_{n}=4n+3$
12. $t\_{n}=2n+1$
13. $t\_{n}=3^{n-1}$
14. $t\_{n}=2∙3^{n}$
15. $t\_{n}=\frac{\left(-2\right)^{n}}{8}$
16. Find the next two terms of each sequence by using the pattern in the differences between terms.
17. 60, 48, 38, 30, 24, \_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
18. 24, 23, 21, 17, 9, \_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
19. 1, 3, 7, 15, 31, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
20. 0, 1, 4, 13, 40, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
21. 1, 1, 2, 3, 5, 8, 13, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_
22. 1, 3, 6, 11, 19, 31, \_\_\_\_\_\_\_\_\_\_ , \_\_\_\_\_\_\_\_\_\_ (Hint: Look at the second differences, that is, the differences of the differences between terms.)

Arithmetic Sequences

 In this section we shall see how we can write and use formulas for arithmetic sequences.

**Consider** the sequence 3, 10, 17, 24, … Verify this is arithmetic and identify the common difference.

 t1 = 3

 t2 =

 t3 =

 t4 =

 tn =

**Result**: For an arithmetic sequence, the formula is tn = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Break for Practice:**

1. Write a formula for each of the following.
2. 7, 15, 23, 31, …
3. 100, 90, 80, 70, …
4. 3, 7, 11, 15, …
5. Find the specified term in the Arithmetic Sequence.
6. 2, 5, 8, … t17 = \_\_\_\_\_\_\_\_\_\_
7. 912, 882, 852, 822, … t43 = \_\_\_\_\_\_\_\_\_\_
8. t2 = 9 t5 = 21 t41 = \_\_\_\_\_\_\_\_\_\_
9. t10 = 41 t15 = 61 t3 = \_\_\_\_\_\_\_\_\_\_

**Extended Practice**:

1. Write a formula for each of the following.
2. 24, 32, 40, 48, …
3. 30, 20, 10, 0, …
4. -3, -10, -17, -24, …
5. -6, -1, 4, 9, …
6. 7, 11, 15, 19, …
7. Find the specified term of each arithmetic sequence.
8. 4, 9, 14, 19, … t21 = \_\_\_\_\_\_\_\_\_\_
9. 3, 11, 19, … t31 = \_\_\_\_\_\_\_\_\_\_
10. 100, 98, 96, … t25 = \_\_\_\_\_\_\_\_\_\_
11. 3, 3.5, 4, 4.5, … t101 = \_\_\_\_\_\_\_\_\_\_
12. 17, 7, -3, … t1000 = \_\_\_\_\_\_\_\_\_\_
13. t2 = 7 t4 = 8 t1 = \_\_\_\_\_\_\_\_\_\_
14. t8 = 60 t12 = 48 t40 = \_\_\_\_\_\_\_\_\_\_

 Now let’s look at a couple of other types of problems that we can solve with the arithmetic sequence formula.

**Break for Practice:**

1. Find the position, n of the underlined term in each arithmetic sequence.
2. 5, 8, 11, 14, … , 68 , …
3. 88, 83, 78, 73, … , 13 , …

 The other idea we will work on is that of finding arithmetic means.

**Arithmetic Means** – terms between two given terms in an arithmetic sequence.

1. Find the stated number of arithmetic means between the two given terms.
2. Two between 8 and 35.
3. Five between 83 and -25.
4. One between -7.8 and 3.6.

**Extended Practice:**

1. Find the position, n, of the underlined term in each arithmetic sequence.
2. 25, 33, 41, …, 145 ,…
3. 40, 37, 34, …, -29, …
4. Find the stated number of arithmetic means between the two given terms.
5. One between -3 and 7
6. One between 2.3 and 9.1
7. Two between 15 and 45
8. Four between 15 and 45
9. How many terms are in the sequence 18, 24, 30, … , 618?
10. How many terms are in the sequence 44, 36, 28, … , -380?

Geometric Sequences

 Now that we have spent time with arithmetic sequences, we will switch our focus to geometric sequences.

**Consider** the sequence 3, 6, 12, 24, … Verify this is geometric and identify the common ratio.

 t1 = 3

 t2 =

 t3 =

 t4 =

 tn =

**Result**: For a geometric sequence, the formula is tn = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Break for Practice:**

1. Write a formula for each of the following.
2. 1000, 200, 40, 8, …
3. -100, 50, -25, 12.5, …
4. Find the specified term in each geometric sequence.
5. 2048, 1024, 512, … t20 = \_\_\_\_\_\_\_\_\_\_
6. 6, 9, 13.5, … t10 = \_\_\_\_\_\_\_\_\_\_
7. t2 = 6 t7 = 192 t12 = \_\_\_\_\_\_\_\_\_\_\_\_

**Extended Practice:**

1. Write a formula for each of the following.
2. 2, 6, 18, 54, …
3. 500, 100, 20, 4, …
4. 64, -48, 36, -27, …
5. Find the specified term of each geometric sequence.
6. 2, 6, 18, 54, … t10 = \_\_\_\_\_\_\_\_\_\_
7. 5, 10, 20, 40, … t12 = \_\_\_\_\_\_\_\_\_\_
8. 40, -20, 10, -5, … t11 = \_\_\_\_\_\_\_\_\_\_
9. -10, 50, -250, 1250, … t9 = \_\_\_\_\_\_\_\_\_\_
10. t2 = 18 t3 = 12 t5 = \_\_\_\_\_\_\_\_\_\_
11. t3 = -12 t6 = 96 t9 = \_\_\_\_\_\_\_\_\_\_

 Now let’s look at a couple of other types of problems that we can solve with the geometric sequence formula.

**Break for Practice:**

1. Find the position, n, of the underlined term in each geometric sequence.
2. $\frac{1}{9} , \frac{1}{3} , 1 , 3,$ … , 19683 , …
3. 17, 34, 68, 136, … , 34,816 , …

 The other idea we will work on is that of finding geometric means.

**Geometric Means** – terms between two given terms in a geometric sequence.

1. Find the stated number of geometric means between the two given terms.
2. Four between 4 and 972.
3. Three between 3 and 48.

**Extended Practice:**

1. Find the position, n, of the underlined term in each geometric sequence.
2. 5, -25, 125, -625, …, 3125 ,…
3. 27, 9, 3, …, $\frac{1}{81}$, …
4. Find the stated number of geometric means between the two given terms.
5. One between 2 and 8
6. One between -18 and -36
7. Three between 5 and 80
8. Two between -4 and 108
9. Tell whether each sequence is arithmetic or geometric. Then find a formula for the sequence.
10. The sequence of positive even integers.
11. 25, 33, 41, 49, …
12. 200, -100, 50, -25, …

Story Problems with Sequences

 In this section we shall see arithmetic and geometric sequences applied to various story problems. You will need to decide if the situation described is arithmetic or geometric in order to solve it.

**Break for Practice:**

1. A part time teacher takes a position at $6,600 per year. He receives annual increases of $250. What will his salary be during his fifteenth year of service?
2. A wealthy man gave his son $5 on his tenth birthday and decided to double his gift each following year. How much did the boy receive on his 21st birthday?
3. A new house purchased for $125,000 is expected to increase in value by 3 % per year. What should its value be in 12 years?
4. A well drilling firm charges $0.35 to drill the first foot, $0.38 for the second foot, and so on in an arithmetic progression. At this rate, how much does the firm charge to drill the last foot of a well 350 feet deep?

**Extended Practice:**

1. Allysa has taken a job with a starting salary of $17,600 and annual raises of $850. What will be her salary during her fifth year on the job?
2. Frank has taken a job with a starting salary of $15,000 and annual raises of 4%. What will be his salary during his third year on the job?
3. An advertisement for a mutual fund claims that people who invested in the fund 5 years ago have doubled their money. If the fund’s future performance is similar to its past performance, how much would a $2,000 investment be worth in 40 years?
4. A culture of yeast doubles in size every 4 hours. If the yeast population is estimated to be 3 million now, what will it be one day from now?

Series and Sigma (Summation) Notation

 Now that we have spent several days exploring sequences, we are ready to explore the topic of series. First we need to understand what a series is.

**Example**: Sequence : 1, 3, 5, 7, 9, …

 Related Series: 1+ 3 + 5 + 7 + 9 + …

**Definition**: A **Series** is the \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .

 Now, since series have an infinite number of terms, most of them will have an infinite sum. Because of this, it is usually more interesting to consider partial sums.

**Definition**: A **Partial Sum** is \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .

Notation: Sn stands for the nth partial sum, which is the sum of the first n terms in a series.

**Example**: Consider 2 + 7 + 12 + 17 + …

 S1 = \_\_\_\_\_\_\_\_\_\_

 S2 = \_\_\_\_\_\_\_\_\_\_

 S3 = \_\_\_\_\_\_\_\_\_\_

 Since it can take a lot of space and time to write out all of the terms, a shorthand notation was developed. This is called sigma or summation notation.

**Sigma or Summation Notation**:

 $S\_{n}=\sum\_{k=1}^{n}t\_{k}$

**Break for Practice**:

1. Expand $S\_{5}=\sum\_{k=1}^{5}(5k+3)$
2. Expand $S\_{4}=\sum\_{k=1}^{4}128\left(\frac{1}{2}\right)^{k-1}$

**Extended Practice:** Expand each of the following partial sums.

1. $\sum\_{k=1}^{6}(k+10)$
2. $\sum\_{k=1}^{8}3k$
3. $\sum\_{k=1}^{6}2^{k}$
4. $\sum\_{k=4}^{10}(3k-2)$
5. $\sum\_{k=0}^{5}\frac{\left(-1\right)^{k}}{k+1}$
6. $\sum\_{k=0}^{3}4^{-k}$
7. $\sum\_{k=3}^{8}\left|5-k\right|$
8. $\sum\_{k=1}^{4}\left(-k\right)^{k+1}$

 Now we will try to go in the reverse direction. We will rewrite a series from expanded form into sigma notation. It will be useful on many of the problems to remember the formulas for arithmetic and geometric sequences.

**Review**: **Arithmetic Sequence** formula: \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

 **Geometric Sequence** formula: \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Break for Practice**: Rewrite each series into sigma notation.

1. 3 + 10 + 17 + 24 + … + 66
2. 3 + 12 + 48 + 192 + … + 12,582,912
3. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+…+\frac{1}{17}$
4. 3 + 9 + 27 + 81 + …
5. -5 + 10 – 20 + 40 - …
6. -6 + 10 – 14 + 18 – 22 + …

**Extended Practice:** Rewrite each series into sigma notation.

1. 2 + 4 + 6 + …+ 1000
2. 5 + 10 + 15 + … + 250
3. $1^{3}+2^{3}+3^{3}+…+20^{3}$
4. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+…+\frac{99}{100}$
5. 3 + 7 + 11 + 15 + … + 399
6. 1 + 2 + 4 + 8 + … + 64
7. -9 + 3 – 1 + $\frac{1}{3}-…$
8. 8 – 4 + 2 – 1 + …
9. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+…$

Sums of Arithmetic Series

 For certain types of series, it is possible to avoid the brute force method of finding partial sums.

Let’s see if we can find a shortcut for arithmetic series.

**Example**: Find $S\_{100}=\sum\_{k=1}^{100}\left(2+5k\right)$

**Result: General Formulas for an Arithmetic Series**

 $S\_{n}=$ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

 $S\_{n}=$ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Break for Practice:** Calculate each partial sum.

1. n = 40 t1 = 2 t40 = 197
2. $\sum\_{k=1}^{30}(4k+3)$
3. 8 + 11 + 14 + … + 308
4. Remember the well drilling problem from earlier in this unit? The company charged $0.35 for the first foot, $0.38 for the second foot, $0.41 for the third foot, etc. What is the total cost for a 350 foot well?

**Extended Practice:** Calculate each partial sum.

1. n = 20 t1 = 5 t20 = 62
2. n = 100 t1 = 17 t100 = 215
3. $\sum\_{k=1}^{100}5k$
4. $\sum\_{m=10}^{20}(30-m)$
5. $\sum\_{j=1}^{50}(3j+2)$
6. The first 100 terms of the series 4 + 7 + 10 + 13 + …
7. 11 + 15 + 19 + … + 83
8. The front row of a theater has 25 seats. Each of the other rows has two more seats than the row before it. How many seats are there altogether in the first 20 rows?
9. Kristen is given a test consisting of 15 questions. The first question is worth five points, and each question after the first is worth three points more than the question before it. What is the maximum score that Kristen can obtain?

Sums of Geometric Series

 There also exists a formula for making the calculation of a partial sum of a geometric series much easier than brute force, too. Let’s try to find that formula.

**Example**: Find $S\_{20}=\sum\_{k=1}^{20}2\left(3\right)^{k-1}$

**Result: General Formula for a Geometric Series**

 $S\_{n}=$

**Break for Practice:** Calculate each partial sum.

1. n = 12 t1 = 2 r = 3
2. $\sum\_{k=1}^{15}8\left(\frac{1}{2}\right)^{k-1}$
3. S12 for 1 – 5 + 25 – 125 + …
4. Recall the wealthy father who gave his son $5 on his tenth birthday. Each year he doubled his gift. What was the total amount given by the son’s 21st birthday?

**Extended Practice:** Calculate each partial sum.

1. n = 8 t1 = 1 r = 2
2. n = 10 t1 = 1 r = -2
3. $\sum\_{k=1}^{12}2^{-k}$
4. $\sum\_{j=1}^{10}\left(-\frac{1}{2}\right)^{j}$
5. Find S10 for the series 24 + 12 + 6 + …
6. Find S20 for the series 1 + 1.1 + 1.21 + …
7. Kurt can trace his ancestors back through 10 generations. He counts his parents as the first generation back, his four grandparents as the second generation back, and so on. How many total ancestors does he have in these 10 generations?
8. You have won a contest sponsored by a local radio station. If you are given the choice of the two payment plans listed below, which plan will pay you more? How much more?

 Plan A: $1 on the first day, $2 on the second day, $3 on the third day, and so on for two weeks.

 Plan B: $0.01 on the first day, $0.02 on the second day, $0.04 on the third day, and so on for two weeks.

Infinite Geometric Series

 Consider the following two geometric series, and calculate the information asked for.

1. 2 + 6 + 18 + 54 + … B) $2+ \frac{2}{3}+\frac{2}{9}+\frac{2}{27}+…$

 r = \_\_\_\_\_\_\_\_\_\_ r = \_\_\_\_\_\_\_\_\_\_

 t1 = \_\_\_\_\_\_\_\_\_\_ t1 = \_\_\_\_\_\_\_\_\_\_

 S1 = \_\_\_\_\_\_\_\_\_\_ S1 = \_\_\_\_\_\_\_\_\_\_

 S5 = \_\_\_\_\_\_\_\_\_\_ S5 = \_\_\_\_\_\_\_\_\_\_

 S10 = \_\_\_\_\_\_\_\_\_\_ S10 = \_\_\_\_\_\_\_\_\_\_

 S50= \_\_\_\_\_\_\_\_\_\_ S50 = \_\_\_\_\_\_\_\_\_\_

**Definitions**: A series \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ if its infinite sum, S, goes to infinity or two different alternating values.

 A series \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ if its infinite sum, S, goes to one specific finite number.

 Which of the two examples converged?

 Which of the two examples diverged?

What part of the formula decides if an infinite geometric series will converge or diverge?

**Summary**: A geometric series will converge if \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ . If it converges the

 limit or sum is found by the formula S =

**Break for Practice:** Decide if the following converge or diverge. If it converges, find the sum.

1. 625 + 125 + 25 + …
2. 15 – 4.5 + 1.35 - …
3. 2 – 3 + 4.5 - …
4. $\sum\_{k=1}^{\infty }2\left(\frac{4}{5}\right)^{k}$
5. A super ball rebounds 95% of the distance it falls. This ball is thrown 12 m in the air (so that the initial up-and-down distance traveled is 24 m). What is the total vertical distance traveled by the ball before it stops bouncing?

**Extended Practice:** Decide if the following converge or diverge. If it converges, find the sum.

1. 24 + 12 + 6 + 3 + …
2. 24 – 12 + 6 – 3 + …
3. 27 – 18 + 12 – 8 + …
4. 27 + 18 + 12 + 8 + …
5. 256 + 320 + 400 + 500 + …
6. 500 + 400 + 320 + 256 + …
7. 3 + 4 + $5\frac{1}{3}+7\frac{1}{9}+…$
8. $\sum\_{n=1}^{\infty }3\left(\frac{1}{4}\right)^{n}$
9. $\sum\_{n=1}^{\infty }\frac{2^{n}}{5^{n}}$
10. A child on a swing is given a big push. She travels 12 feet on the first back-and-forth swing but only $\frac{5}{6}$ as far on each successive back-and-forth swing. How far does she travel before the swing stops?