**Algebra II**

**Unit 8**

**Polynomial Equations and Variations**

Unit “I can” statements:

1. I can divide polynomials using long division.
2. I can divide polynomials using synthetic division.
3. I can use the remainder and factor theorems to find factors of polynomials and to solve polynomial equations.
4. I can apply the rational root theorem to find all of the rational roots of a polynomial function.
5. I can draw appropriate graphs for higher degree polynomial functions when given key facts about the functions.
6. I can find all rational and complex zeros of higher degree polynomial functions.
7. I can solve problems using direct, inverse, and joint variations.
8. I can use linear interpolation to find values not listed in a given table of data.

Common Core State Standards that are addressed in this unit include: A.CED.2a, A.CED.4a, A.APR.2b, A.APR.6d, N.CN.7c, N.CN.9c, F.IF.8

For more information see [www.corestandards.org/Math/](http://www.corestandards.org/Math/)

Dividing Polynomials with Long Division

 In this unit we will be exploring higher degree polynomial equations and functions. In order to solve these equations or graph these functions, it is often necessary to be able to identify the factors of the polynomial. One way to check if a number/expression is a factor of another is by using division. A factor will leave a remainder of zero.

 In this unit we will learn two different ways to divide polynomials. Understanding the situation will help you to decide which method to apply.

 The first method that we will spend time with is long division. The chief pro of the long division method is that it will work in all situations with all types of polynomials. The process is similar to normal long division with plain numbers.

**Review**: Divide 2560 by 12 using long division.

Now we will apply the same process to polynomials. **Note**: Before using long division, always write both polynomials in descending order, and insert any “missing terms” by using a coefficient of zero.

**Break for Practice**: Divide

1. $\frac{3x^{3}-2x^{2}-13x+10}{x-2}$ 2. $\frac{x^{3}-4x^{2}-19x+9}{x+3} $
2. $\frac{x^{3}+8}{x+2}$ 4. $\frac{6x^{3}-19x^{2}+15}{3x-5}$
3. $\frac{x^{4}-x^{3}+7x+5}{x^{2}+2x+1}$

**Extended Practice**: Divide by using long division.

1. $\frac{x^{2}+3x-4}{x+2}$ 2. $\frac{x^{2}-x+3}{x+1}$

3. $\frac{9z-z^{2}}{z-3}$ 4. $\frac{x^{3}-x^{2}-10x+10}{x-3}$

5. $\frac{4t^{2}-4t+1}{2t+1}$ 6. $\frac{6u^{2}+7u+5}{3u-1}$

7. $\frac{2s^{3}-29s+13}{s+4}$ 8. $\frac{15z^{3}-z^{2}-11z-3}{3z^{2}-2z-1}$

Dividing Polynomials with Synthetic Division

 In this section, we will learn how to divide polynomials using a technique called synthetic division. The pros of this method include its speed and compactness. The con is that it can only be used when dividing by polynomials in the form $\pm b$ . Remember to write all polynomials in descending order and insert any “missing terms.”

**Example**: Divide with synthetic division.

$$\frac{3x^{3}-2x^{2}-13x+10}{x-2}$$

**Break for Practice:** Divide by using synthetic division

1. $\frac{x^{2}+3x-4}{x+2}$ 2. $\frac{x^{3}+8}{x+2}$

3. $\frac{6x^{3}-7x^{2}+3x+2}{3x+1}$

1. $\frac{6t^{3}+t^{2}+7t+10}{3t+2}$

**Extended Practice**: Divide by using synthetic division.

1. $\frac{3x^{3}-5x^{2}+x-2}{x-2}$ 2. $\frac{x^{3}+3x^{2}-2x-6}{x+3}$

3. $\frac{t^{4}+5t^{3}-2t-7}{t+5}$ 4. $\frac{2s^{4}-7s^{3}+7s+6}{s-3}$

1. $\frac{x^{5}-1}{x-1}$ 6. $\frac{2x^{4}+x^{3}-x-2}{x+1}$

7. $\frac{2x^{3}-3x^{2}+4x-2}{2x+1}$ 8. $\frac{6t^{4}+5t^{3}-10t+4}{3t-2}$

The Remainder and Factor Theorems

 In this section we will look at two closely related theorems that will aid us when we begin factoring and graphing higher degree polynomial functions.

**Consider**: $P\left(x\right)=x^{2}-5x+6$

 Evaluate $P(1)$ Now try this: 1 1 -5 6

Hmm…

 Evaluate $P(2)$ Now try this: 2 1 -5 6

Hmm…

 From these examples, we can see that the remainder in synthetic division can also be used to find the value of a function at x. If the remainder is zero, then the value of the function is zero, and a factor of the polynomial would be in the form (x – that value). These are the ideas stated in the Remainder and Factor Theorems.

**Remainder Theorem**: You can evaluate a polynomial at a certain value by just putting that value in

the box of synthetic division. The \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ in the \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ position is the

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ of the function.

**Factor Theorem**: \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ is a factor of P(x) if and only if P(b) = \_\_\_\_\_\_\_\_\_\_ .

**Break for Practice**:

1. Use synthetic substitution for find P(c).

1. $P\left(x\right)=x^{3}-3x^{2}-4x+6 c=3$
2. $P\left(x\right)=4x^{3}+2x^{2}-4x+6 c=-\frac{3}{2}$
3. Use the factor theorem to determine whether the binomial is a factor of the given polynomial.
4. $x-1 ; P\left(x\right)=x^{6}-x^{4}+x^{2}-1$
5. $y+2 ; P\left(y\right)=y^{4}-y^{2}+4y+2$
6. A root (solution) of the equation is given. Solve the equation.
7. $x^{3}+3x^{2}-18x-40=0 ; 4$
8. $x^{3}-5x-2=0 ; -2$

**Extended Practice**

1. Use synthetic substitution for find P(c).

1. $P\left(x\right)=x^{3}-2x^{2}-5x-7 c=4$
2. $P\left(x\right)=2x^{3}+3x^{2}-5x+2 c=-3$
3. $P\left(x\right)=4x^{3}-4x^{2}+5x+1 c=\frac{3}{2}$
4. $P\left(x\right)=2x^{4}-x^{3}+x-2 c=-\frac{3}{2}$
5. Use the factor theorem to determine whether the binomial is a factor of the given polynomial.
6. $x+1 ; P\left(x\right)=x^{7}-x^{5}+x^{3}-x$
7. $y+1; P\left(y\right)=y^{5}+y^{4}+y^{3}+y^{2}+y+1$
8. $z+2 ; P\left(z\right)=z^{5}+2z^{4}+z^{3}+2z^{2}+z+2$
9. A root (solution) of the equation is given. Solve the equation.
10. $x^{3}+3x^{2}-3x-9=0 ; -3$
11. $2x^{3}+9x^{2}+7x-6=0 ; -2$

The Rational Root Theorem

 In this section we will use an extension of the factor theorem, the graphing calculator/computer, and synthetic substitution to factor higher degree polynomials that have all rational roots.

**Rational Root Theorem**: $\left(ax-b\right) is a factor of P\left(x\right) if and only if P\left(\frac{b}{a}\right)=0$.

 Note: The maximum number of roots is equal to the degree of the polynomial.

**Break for Practice**:

1. Factor $2x^{3}-3x^{2}-8x-3$
2. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
3. List all of the possible rational roots.
4. Sketch a graph.
5. Write the list of factors.

1. Factor $x^{4}-5x^{2}+4$
2. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
3. List all of the possible rational roots.
4. Sketch a graph.
5. Write the list of factors.

**Extended Practice**:

1. Factor $x^{3}+2x^{2}-5x-6$
2. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
3. List all of the possible rational roots.
4. Sketch a graph.
5. Write the list of factors.
6. Factor $x^{3}-19x+30$
7. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
8. List all of the possible rational roots.
9. Sketch a graph.
10. Write the list of factors.
11. Factor $2x^{3}+5x^{2}-4x-3$
12. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
13. List all of the possible rational roots.
14. Sketch a graph.
15. Write the list of factors.
16. Factor $4x^{4}-11x^{3}-7x^{2}+11x+3$
17. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
18. List all of the possible rational roots.
19. Sketch a graph.
20. Write the list of factors.
21. Factor $4x^{4}+8x^{3}+3x^{2}-2x-1$
22. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
23. List all of the possible rational roots.
24. Sketch a graph.
25. Write the list of factors.
26. Factor $x^{5}+2x^{4}-10x^{3}-20x^{2}+9x+18$
27. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
28. List all of the possible rational roots.
29. Sketch a graph.
30. Write the list of factors.
31. Factor $x^{5}+15x^{4}+85x^{3}+225x^{2}+274x+120$
32. How many factors should we expect? \_\_\_\_\_\_\_\_\_\_
33. List all of the possible rational roots.
34. Sketch a graph.
35. Write the list of factors.

Some Useful Theorems

 In the last section, we learned how to find all rational solutions to polynomial equations. In this section we shall expand the type of solutions that we are able to find.

**Example**: Solve the equation $x^{2}-2x+5=0$

What did you notice about the solutions to the above equation?

**Theorem**: If a polynomial has \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ coefficients, then if there is a \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

solution, there will actually be a \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ of \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ solutions, and they will be

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.

There is another important theorem that states:

**Theorem**: A \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ equation of degree n , will have \_\_\_\_\_\_\_\_\_\_ solutions. (These include real, imaginary, complex, and duplicates.)

**Break for Practice**:

1. Find all of the solutions for $x^{3}=27.$
2. Find all of the solutions for $x^{4}=16.$
3. Find a second degree equation with solutions of x = 3, and x = -4.
4. Find a third degree equation with solutions x = 3*i*, and x = 5.

**Extended Practice**:

1. Solve the following equations.
2. $x^{2}+3x+5=0$ b) $x^{2}-3x+6=0$ c) $-x^{2}-2x-6=0$
3. Find all of the solutions for the following equations.
4. $x^{3}=8$ b) $x^{3}=125$

c) $x^{4}=16$ b) $x^{4}=81$

1. Write a second-degree equation which has solutions of x = 5, and x = -2.
2. Write a third degree equation which includes solutions of x = 4*i*, and x = 5.

Graphing Higher Degree Polynomial Functions

 In this section we will be working with the graphs of higher degree polynomial functions. You will be looking for patterns in the shapes of the graphs. You will also learn how to tell how many zeros are real, and how many are imaginary. First you need to learn a little background.

For each graph tell what you notice about the direction of the “tails”, how many bumps (relative max and min) that you see, and the number of real zeros that are on the graph.







**Break for Practice**: For each polynomial, draw the graph and write the answers to the following questions.

**Questions**: a) What is the direction of the tails?

 b) What is the degree of the polynomial?

 c) How many bumps (relative max and min) are in the graph?

 d) How many real zeros are there?

 e) How many imaginary zeros are there?

$$y=-x^{2}+8x-16$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=2x^{2}+5x-3$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=-\frac{2}{3}x-5$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=3x+4$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{4}-x^{3}+3x^{2}-9x-54$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{4}-2x^{3}-13x^{2}+14x+24$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{3}-5x^{2}+3x+9$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{3}-3x^{2}+4x-12$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{3}+2x^{2}-5x-6$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{2}+3x+5$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

**Questions**:

1. What is true about the direction of the tails in an odd degree function?
2. What is true about the direction of the tails in an even degree function?
3. How does the maximum number of bumps (relative max and min) compare to the degree of a function?

$$y=x^{6}+2x^{5}-3x^{4}+84x^{3}-184x^{2}+304x-624$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{5}+3x^{4}-7x^{3}-11x^{2}+6x+8$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{5}-2x^{4}+40x^{2}-41x-78$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

$$y=x^{6}-14x^{4}+49x^{2}-36$$

a) tails?

b) degree?

c) bumps?

d) real zeros?

e) imaginary zeros?

**Extended Practice**: Use what you have observed about the graphs of higher degree functions to sketch graphs of the functions described.

1. Quintic (5th degree) function with exactly 3 real zeros
2. Sixth degree function with exactly 4 real zeros
3. Cubic function with exactly two distinct real zeros
4. Quartic (4th degree) function with no real zeros
5. Cubic function with no real zeros
6. Quartic function with exactly five real zeros

Finding all Rational and Complex Zeros of Polynomial Functions

 In this section we will take all that we have previously learned to find all of the rational and complex zeros of a polynomial function.

**Break for Practice**: Find all of the rational and complex zeros for each polynomial function.

1. $y=x^{3}+x^{2}-6x$
2. $y=x^{4}-x^{3}-3x^{2}-9x-108$
3. $y=x^{4}-6x^{3}+14x^{2}-16x+8$

**Extended Practice**: Find all of the zeros for each polynomial function.

1. $y=x^{3}-3x^{2}+4x-12$
2. $y=x^{3}-5x^{2}+3x+9$
3. $y=x^{4}-x^{3}+3x^{2}-9x-54$
4. $y=x^{5}-2x^{4}+40x^{2}-41x-78$
5. $y=-2x^{6}+5x^{5}+34x^{4}-16x^{3}-209x^{2}-376x+240 $

Try a viewing window of x-min = -5, x-max = 5, y-min = -1500, and y-max = 1000

Direct Variation and Proportion

 To finish this unit, we will look at different types of variations.

**Definition**: A **Direct Variation** is a function in the form \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

 Notes: As x increases, so does y.

 k is the constant of variation or proportion

 n = 1 unless told differently

**Example**: If y varies directly as x, and y = 12 when x = 20, find y when x = 50.

 Method 1 Method 2

**Break for Practice**: Solve using either method.

1. If y varies directly as x, and y = 6 when x = 4, find y when x = 12.
2. If w varies directly as z, and w = 4.5 when z = 3, find z when w = 1.5.
3. If p is directly proportional to q3  , and p = 3 when q = 2, find p when q = 4.
4. If x varies directly as 3y+2, and x = 10 when y = 6, find y when x = 7.
5. If the sales tax on a $38 purchase is $2.85, what will the tax be on an $84 purchase?
6. A survey showed that 52 out of 234 people questioned preferred hot cereal to cold. In a school of 1800 people, how many people are likely to prefer hot cereal?

**Extended Practice**: Solve using either method

1. If y varies directly as x, and y = 6 when x = 15, find y when x = 25.
2. If s is directly proportional to t, and s = 40 when t = 15, find t when s = 64.
3. If p is directly proportional to q, and p = 9 when q = 7.5, find q when p = 24.
4. If s varies directly as r2 , and s = 12 when r = 2, find s when r = 5.
5. If y is directly proportional to $\sqrt{x}$ , and y = 25 when x = 3, find x when y = 100.
6. If w varies directly as 2x – 1, and w = 9 when x = 2, find x when w = 15.
7. If the sales tax on a $60 purchase is $3.90, what would it be on a $280 purchase?
8. A real estate agent made a commission of $5400 on a house that sold at $120,000. At this rate, what commission will the agent make on a house that sells for $145,000?
9. The acceleration of an object varies directly as the force acting on it. If a force of 240 newtons causes an acceleration of 150 m/s2, what force will cause an acceleration of 100 m/s2.
10. A public-opinion poll found that of a sample of 450 voters, 252 favored a school bond measure. If 20,000 people vote, about how many are likely to vote for the bond measure?

Inverse and Joint Variations

 In the last section we learned what a direct variation is. Now we will learn what an inverse variation is.

**Definition**: An **Inverse Variation** is a function in the form \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,

 Note: As x increases, y will decrease.

**Example**: If y varies inversely as x, and y = 5 when x = 4, find x when y = 10.

**Break for Practice**: Solve

1. If a is inversely proportional to b, and b = 12 when a = 8, find b when a = 3.
2. If x varies inversely as the square of y, and x = 2 when y = 12, find y when x = 8

 It is possible to work with more than one variation at a time. If the term **Jointly** is used, then it means that several variables are varying **directly**.

**Break for Practice**: Solve

1. If x varies jointly as y and z, and x = 100 when y = 20 and z = 10, find x when y = 60 and z = 30.
2. If x is jointly proportional to y and the square root of z, and x = 20 when y = 5 and z = 9, find x when y = 6 and z = 25.
3. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. 100 m of wire with a diameter of 6 mm has a resistance of 12 ohms. Eighty meters of a second wire of the same material has a resistance of 15 ohms. Find the diameter of the second wire.

**Extended Practice**: Solve

1. If y varies inversely as x, and y = 3 when x = 6, find x when y = 18.
2. If z is inversely proportional to r, and z = 32 when r = 1.5, find r when z = 8.
3. If w is inversely proportional to the square of v, and w = 3 when v = 6, find w when v = 3.
4. If p varies inversely as the square root of q, and p= 12 when q = 36, find p when q = 16.
5. If z is jointly proportional to x and y, and z = 18 when x = 0.4 and y = 3, find z when x = 1.2 and y = 2.
6. If w is jointly proportional to u and v, and w = 24 when u = 0.8 and v = 5, find u when w = 18 and v = 2.
7. If s varies directly as r and inversely as t, and s = 10 when r = 5 and t = 3, for what value of t will s = 3 when r = 4?
8. Suppose that r varies directly as p and inversely as q2, and that r = 27 when p = 3 and q = 2. Find r when p= 2 and q = 3.
9. The frequency of a radio signal varies inversely as the wave length. A signal of frequency 1200kilohertz (kHz), which might be the frequency of an AM radio station, has wave length 250 m. What frequency has a signal of wave length 400m?
10. The stretch in a wire under a given tension varies directly as the length of the wire and inversely as the square of its diameter. A wire having length 2 m and diameter 1.5 mm stretches 1.2 mm. If a second wire of the same material (and under the same tension) has length 3 m and diameter 2.0 mm, find the amount of stretch.

Linear Interpolation

 We will finish this unit with the topic of linear interpolation. Linear interpolation is a method to estimate values that are not given in a table or chart. It makes use of ratios and proportions.

**Break for Practice**:

1. Use linear interpolation and the table to find these values to the nearest integer.
2. Find the approximate number of bachelor’s degrees that were awarded in computer science in 1971?

|  |  |
| --- | --- |
| **Year** | **Bachelor’s degrees awarded in computer science** |
| 1968 | 459 |
| 1972 | 3,402 |
| 1976 | 5,700 |
| 1980 | 11,154 |
| 1984 | 32,172 |

1. Find the approximate year that 10,000 bachelor’s degrees were awarded in computer science.
2. Use linear interpolation and the table to find these values to the nearest integer.

The table gives the temperature in degrees Fahrenheit on a spring day in Boston, MA.

|  |  |
| --- | --- |
| **Time (pm)** | **Temperature (F)** |
| 1:00 | 68 |
| 2:00 | 66 |
| 3:00 | 63 |
| 4:00 | 59 |
| 5:00 | 53 |
| 6:00 | 45 |
| 7:00 | 39 |

1. Approximate the temperature at 3:40 pm.
2. At about what time was the temperature 40º?

**Extended Practice**: Solve each problem using linear interpolation.

1. Consider the table of population figures for the following questions.
2. Approximate the population in 1915.

|  |  |
| --- | --- |
| Year | U.S. Population in millions |
| 1900 | 76 |
| 1910 | 92 |
| 1920 | 106 |
| 1930 | 123 |
| 1940 | 132 |
| 1950 | 151 |
| 1960 | 179 |
| 1970 | 203 |
| 1980 | 227 |
| 1990 | 243 |

1. Approximate the population in 1963.
2. Approximate the year that the population was 100 million.
3. Approximate the year that the population was 200 million.
4. The table gives the density of dry air at various altitudes.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Altitude (m) | 0 | 500 | 1000 | 1500 | 2000 | 2500 | 3000 | 3500 |
| Density (kg/m3) | 1.225 | 1.167 | 1.112 | 1.058 | 1.007 | 0.957 | 0.909 | 0.863 |

1. Approximate the density at an altitude of 1200 m.
2. Approximate the density at an altitude of 3200 m.
3. Approximate the altitude for dry air with a density of 1.200 kg/m3.
4. Approximate the altitude for dry air with a density of 0.930 kg/m3.